Journal of Computational and Applied Mathematics 309 (2017) 396–407

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Computing probabilistic solutions of the Bernoulli random differential equation

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ARTICLE INFO

Article history:
Received 25 September 2015
Received in revised form 4 December 2015

Keywords:
Bernoulli random differential equation
First probability density function
Probabilistic solution
Random variable transformation technique

ABSTRACT

The random variable transformation technique is a powerful method to determine the probabilistic solution for random differential equations represented by the first probability density function of the solution stochastic process. In this paper, that technique is applied to construct a closed form expression of the solution for the Bernoulli random differential equation. In order to account for the general scenario, all the input parameters (coefficients and initial condition) are assumed to be absolutely continuous random variables with an arbitrary joint probability density function. The analysis is split into two cases for which an illustrative example is provided. Finally, a fish weight growth model is considered to illustrate the usefulness of the theoretical results previously established using real data.

1. Introduction and motivation

Ever since the early contributions by I. Newton, G.W. Leibniz, Jacob and Johann Bernoulli in the XVII century until now, differential equations have uninterruptedly demonstrated their capability to model successfully complex problems. There is virtually no applied scientific area where differential equations had not been used to deal with relevant problems. Numerous examples can be found in engineering, physics, chemistry, epidemiology, economics, etc. From a practical standpoint, the application of differential equations requires setting their inputs (coefficients, source term, initial and boundary conditions) using sampled data, thus containing uncertainty stemming from measurement errors. It has led to the consideration of randomness in the formulation of continuous models based on differential equations. In this regard, there are two main classes of equations, stochastic differential equations and random differential equations. In the former case, differential equations are forced by an irregular process such as a Wiener process or Brownian motion. This class of equations are usually written in terms of stochastic differentials, although they are interpreted as Itô stochastic integrals [1,2]. Solutions of Itô-type stochastic differential equations typically exhibit nondifferentiability of their sample paths or trajectories due to the irregularity of the driving Brownian motion. The formulation of Itô-type stochastic differential equations from their deterministic counterpart can be justified by means of the perturbation of input parameters via white noise process, i.e., the formal derivative of a Brownian motion. This implicitly entails that Gaussian-type uncertainty is assumed for perturbed inputs. Although, stochastic differential equations have demonstrated to be powerful mathematical representations to model many problems, for instance in finance, engineering, biosciences, etc., [3–8], clearly this approach does not cover important casuistries. A complementary approach to introduce uncertainty in differential equations is to allow the direct

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http://dx.doi.org/10.1016/j.cam.2016.02.034

assignment of any probability distribution to input parameters, which is referred to as the *randomization process* of the deterministic or classical differential equation. If coefficients are assumed to be random variables (r.v.'s), then, for example, beta, exponential, Gaussian, etc., may be appropriate candidate probability distributions to account for model uncertainty. In principle, this approach is more flexible and it leads to random differential equations. Throughout this paper, random differential equations will considered only.

Similarly to deterministic case, the first goal when dealing with both stochastic and random differential equations is computing, exactly or approximately, the solution stochastic process (s.p.). Unlike deterministic context, now the determination of the main statistical functions associated to the solution s.p., such as the mean and variance functions, are also important goals to be achieved. In fact, the average behaviour of the solution as well as its variability around the mean are obtained from these two statistical moments. Although this information is valuable, and most contributions focus on the computation of the solution s.p. and its mean and variance/standard deviation functions, a more ambitious target includes the determination of the first probability density function (1-p.d.f.) of the solution. The 1-p.d.f. provides a full probabilistic description in each time instant of the solution s.p. Moreover, from the 1-p.d.f., both the mean and variance functions can be straightforwardly computed, but also asymmetry, kurtosis, and other higher unidimensional statistical moments. Even though in this paper we focus on the computation of the 1-p.d.f. of the solution, it is worth underlining that higher p.d.f.'s of the solution s.p. are also useful for giving further statistical characteristics. For example, from the second p.d.f. one can obtain the correlation function of the s.p. which gives a measure of the linear interdependence between the r.v.'s coming from evaluating the s.p. in two different time instants [4, p.39].

In order to determine the 1-p.d.f., the so-called random variable transformation (RVT) method will be applied throughout this paper. RVT method is a powerful technique that permits the computation of the p.d.f. of a r.v. which is obtained after mapping another r.v. whose p.d.f. is given [9]. A generalization of this method can be found in [10]. One of the most fruitful applications of RVT technique is getting the complete probabilistic description of the solution to random differential equations represented by the 1-p.d.f. of the solution s.p. Some recent contributions addressed to determine the 1-p.d.f. of the solution of particular random differential equations can be found, for example, in [11–13]. In [11] one provides a comprehensive study to compute the 1-p.d.f. of the solution s.p. of the random linear first-order differential equation. The study considers all possible cases with respect to the manner that randomness can appear either in the diffusion coefficient, source term or/and the initial condition. In [12] a logistic model where only the initial condition is random. Authors determine the 1-p.d.f. of the proportion of susceptibles of a SI-type epidemiological model. From the 1-p.d.f., a number of probabilistic properties of the solution s.p., such as the mean, the variance, the quartiles, etc., are given. The results obtained in this latter paper have been recently generalized in [13] by assuming that all input parameters are r.v.'s. In these three contributions on random ordinary differential equations, the RVT method constitutes the cornerstone to conduct their respective analyses. However, its applicability goes beyond random ordinary differential equations. For example, some interesting contributions deal with random partial differential equations [14,15]; random integral–differential equations [16] and random difference equations [17]. Although, in all these contributions the 1-p.d.f of the solution s.p. of the corresponding problems is obtained in an exact way, the technique can be also applied to get numerical approximations, [18].

Recently, RVT technique has been applied by the authors to give a full probabilistic description to both, general linear first-order and Riccati random differential equations, represented by the 1-p.d.f. of their solutions [11,19]. The aim of this paper is to continue extending this analysis to another important classical differential equations where probabilistic dependence among input r.v.'s will be assumed. In the following, we will consider the Bernoulli random initial value problem (IVP)

$$\left.\begin{array}{l}\dot{X}(t) = CX(t) + D(X(t))^{A}, \quad t \geq t_0,\\ X(t_0) = X_0,\end{array}\right\} \tag{1}$$

where t_0 denotes the initial time and all the input parameters, X_0, D, C and A, are assumed to be absolutely continuous r.v.'s defined in a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Hereinafter, $\mathcal{D}(X_0)$, $\mathcal{D}(D)$, $\mathcal{C}(C)$ and $\mathcal{D}(A)$ will denote their respectively domains. In order to provide as much generality as possible throughout our analysis, hereinafter we will assume that X_0, C, D and A are statistically dependent. In the following, $f_{X_0,D,C,A}(x_0, d, c, a)$ will denote their joint p.d.f.

The paper is organized as follows. In Section 2, some preliminaries and technical results about RVT technique that will be required throughout the paper, are included. Section 3 is addressed to determine the 1-p.d.f. of the solution s.p. to the Bernoulli random IVP (1) in the general scenario where all input parameters (X_0, D, C, A) are assumed to be r.v.'s. As it will be shown later, our approach requires splitting the analysis in two cases. For every case, an illustrative example is also provided. In Section 4, we take advantage of the ideas exhibited in Section 3 to illustrate the usefulness of computing the 1-p.d.f. to deal with a fish weight growth model. Conclusions are drawn in Section 5.

2. Preliminaries

In this section, some technical results that will play a key role to solve the Bernoulli random IVP (1) are presented.

For the sake of clarity, we start by stating the Random Variable Transformation (RVT) method. This result permits the computation of the p.d.f. of a r.v. which is obtained after transforming another r.v. whose p.d.f. is known.

Theorem 1 (*Multidimensional Random Variable Transformation Method, [4]*)**.** *Let us consider* $\mathbf{U} = (U_1, \ldots, U_n)^{\top}$ *and* $\mathbf{V} = (V_1, \ldots, V_n)^{\top}$ *two n-dimensional absolutely continuous random vectors defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*. Let* $\mathbf{g}$: