Journal of Computational and Applied Mathematics 309 (2017) 482–492

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Geometrical definition of a continuous family of time transformations generalizing and including the classic anomalies of the elliptic two-body problem

José A. López Ortí [a,*], Francisco José Marco Castillo [a], María José Martínez Usó [b]

[a] Departamento de Matemáticas, IMAC, Universidad Jaume I, Av Sos Baynat s/n, E-12071 Castellón, Spain

[b] Departamento de Matemática Aplicada, IUMPA, Universidad Politécnica de Valencia, Camino Vera s/n, E-46022 Valencia, Spain

ARTICLE INFO

Article history:
Received 27 October 2015
Received in revised form 15 February 2016

MSC:
70F05
70F10
70F15
70M20

Keywords:
Celestial mechanics
Orbital motion
Ordinary differential equations
Computational algebra

ABSTRACT

This paper is aimed to address the study of techniques focused on the use of a family of anomalies based on a family of geometric transformations that includes the true anomaly f, the eccentric anomaly g and the secondary anomaly f' defined as the polar angle with respect to the secondary focus of the ellipse.

This family is constructed using a natural generalization of the eccentric anomaly. The use of this family allows closed equations for the classical quantities of the two body problem that extends the classic, which are referred to eccentric, true and secondary anomalies.

In this paper we obtain the exact analytical development of the basic quantities of the two body problem in order to be used in the analytical theories of the planetary motion. In addition, this paper includes the study of the minimization of the errors in the numerical integration by an appropriate choice of parameters in our selected family of anomalies for each value of the eccentricity.

1. Introduction

The study of the motion in the solar system is one of strengths of Celestial Mechanics. This issue involves the development of planetary theories and the motion of artificial satellites around the earth. In this paper, we deal with both topics.

To construct a planetary theory two major ways can be considered: the use of a numerical integrator [1,2] or the use of analytical methods to integrate the problem [3–6].

The analytical methods are based on the solution of the two body problem (Sun-planet) through a set of orbital elements, for example the third set of Brower and Clemence [7] $(a, e, i, \Omega, \omega, M)$, where $M = M_0 + n(t - t_0)$, n is the mean motion, t_0 is the initial epoch whose value are constant in the unperturbed two body problem and M_o the mean anomaly in the initial epoch t_0. This solution can be considered as a first approximation of the perturbed problem and we can use the Lagrange method of variation of constants to replace the first elements by the osculating ones given by the Lagrange planetary equations [8]

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial \sigma}$$
$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial \sigma}$$

* Corresponding author.
E-mail addresses: lopez@mat.uji.es (J.A. López Ortí), marco@mat.uji.es (F.J. Marco Castillo), mjmartin@mat.upv.es (M.J. Martínez Usó).

http://dx.doi.org/10.1016/j.cam.2016.02.041
0377-0427/

$$\frac{di}{dt} = -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} + \frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega}$$
$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}$$
$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}$$
$$\frac{d\sigma}{dt} = -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} \tag{1}$$

σ is a new variable defined by the equation:

$$M = \sigma + \int_{t_0}^{t} n\,dt \tag{2}$$

and it coincides with M_0 in the case of the unperturbed motion. R is the disturbing potential $R = \sum_{k=1}^{N} R_i$ due to the disturbing bodies $i = 1, \ldots, N$. It is defined as [8]

$$R = \sum_{k=1}^{N} Gm_k \left[\left(\frac{1}{\Delta_k} \right) - \frac{x \cdot x_k + y \cdot y_k + z \cdot z_k}{r_k^3} \right] \tag{3}$$

where $\vec{r} = (x, y, z)$ and $\vec{r}_k = (x_k, y_k, z_k)$ are the heliocentric vector position of the secondary body and the kth disturbing body respectively, Δ_k is the distance between the secondary body and the disturbing body, and m_k the mass of the disturbing body.

In order to integrate the Lagrange planetary equations through analytical methods it is necessary to develop the second member of the Lagrange planetary equations as truncated Fourier series, which is a classical problem in celestial mechanics [6,9,7,10,11]. The analytical methods provide very long series solution and it is suitable to obtain more compact developments using as temporal variable an appropriate anomaly.

To obtain the expansions according to an anomaly Ψ_i it is necessary to obtain for each planet i the developments of the coordinates and the inverse of the radius in Fourier series of Ψ_i. Then, the integration of the Lagrange planetary equations with respect to the Ψ_i anomalies requires to compute the corresponding Kepler equation $M_i = M_i(\Psi_i)$ [12–14].

When using numerical integration methods it is more appropriate to consider the equation of motion in the form of the second Newton law. The efficiency of the numerical integrators can be improved through an appropriate change in the temporal variable. In this paper we will study the performance of the previous family of anomalies. To this aim, we select the problem of the motion of an artificial satellite around the Earth. The relative motion of the secondary with respect to the Earth is defined by the second order differential equations

$$\frac{d^2\vec{r}}{dt^2} = -GM\frac{\vec{r}}{r^3} - \vec{\nabla}U - \vec{F} \tag{4}$$

where $\vec{r}$ is the radius vector of the satellite, U the potential from which the perturbative conservative forces are derived and $\vec{F}$ includes the non-conservative forces. To integrate the system (4) it is necessary to know the initial values of the radius vector $\vec{r}_0$ and velocity $\vec{v}_0$.

In order to uniformize the truncation errors when a numerical integrator is used there are three main techniques:

1. The use of a very small stepsize.
2. The use of an adaptative stepsize method.
3. The use of a change in the temporal variable to arrange an appropriate distribution of the points on the orbit so that the points are mostly concentrated in the regions where the speed and curvature are maxima.

This paper follows the third technique. Several authors have already studied this question. See for instance, Sundman [15], who introduced a new temporal variable τ related to the time t through $dt = Crd\tau$, Nacozy [16] proposed a new temporal variable $dt = Cr^{3/3}d\tau$, Brumberg [17] proposed the use of the regularized length of arc and Brumberg and Fukushima [18] introduced the elliptic anomaly as temporal variable. Janin [19,20] and Velez [21] extended this technique defining a new one-parameter family of transformations α called generalized Sundman transformations $dt = Q(r, \alpha)d\tau_\alpha$, where $Q(r, \alpha) = C_\alpha r^\alpha$. The function $Q(r)$ is normally known as partition function. A more complicated family of transformations was introduced by Ferrándiz [22] $Q(r) = r^{2/3}(a_0 + a_1 r)^{-1/2}$. López [23] introduces an new family of anomalies, called natural anomalies as $\Psi_\alpha = (1-\alpha)f' + \alpha f$, $\alpha \in [0, 1]$ where f, f' are the true and secondary anomalies it is the angle between the periapsis and the secondary position taking as origin the primary focus F or the secondary focus of the ellipse f' respectively. Analytical and numerical properties of generalized Sundman anomalies and natural have been studied by López et al. [24–26].