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Third-degree anomalies of Traub's method[☆]

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ABSTRACT

Traub's method is a tough competitor of Newton's scheme for solving nonlinear equations as well as nonlinear systems. Due to its third-order convergence and its low computational cost, it is a good procedure to be applied on complicated multidimensional problems. In order to better understand its behavior, the stability of the method is analyzed on cubic polynomials, showing the existence of very small regions with unstable behavior. Finally, the performance of the method on cubic matrix equations arising in control theory is presented, showing a good performance.

1. Motivation

In many branches of Science and Technology it is necessary to solve different kinds of nonlinear equations or systems $F(x) = 0$, where $F : X \to Y$, being X and Y Banach spaces. The best known iterative scheme is Newton's method

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \quad k = 0, 1, \ldots$$

but Traub's scheme increases the order of convergence of Newton's one, without a complex iterative formula

$$\begin{aligned} y^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - [F'(x^{(k)})]^{-1}F(y^{(k)}), \quad k = 0, 1, \ldots \end{aligned} \tag{1}$$

where $F'(x)$ denotes the Fréchet derivative of F. This scheme can be successfully used, with third-order convergence, on nonlinear problems.

In Control Theory (in the calculation of the logarithm of a matrix or in the computation of sector function), nuclear magnetic resonance, lattice quantum chromo-dynamics and other areas of applications, matrix equations such as $X^p - A = 0$ where the pth root of a matrix A must be calculated, can appear (see, for example, [1–3]). Most of the known algorithms are useless for their numerical instability, unless A is very well conditioned. So, in order to adapt only the best iterative methods

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for solving this kind of nonlinear problems, we wonder about their behavior on these polynomials, as many of them can be adapted to solve matrix equations holding the order of convergence but it is necessary to know about their stability properties.

In the last few years, the use of tools from Complex Dynamics has allowed the researchers in this area of Numerical Analysis to understand the stability of iterative schemes deeply. (see, for example, [4–13]). The analysis, in these terms, of the rational function R associated with the iterative procedure applied on quadratic polynomials, gives us valuable information about its role on the convergence's dependence on initial estimations, the size and shape of convergence regions and even on a possible convergence to fixed points that are not solution of the problems to be solved or to different attracting or even superattracting cycles. Moreover, if a parametric family is studied under this point of view, the most stable elements of the class can be chosen, by means of an appropriated use of the parameter plane.

In this paper, we analyze the dynamics of the rational operator associated to Traub's method on cubic polynomials. Stable and pathological behaviors are obtained depending on the polynomial.

1.1. Dynamical concepts

In this section, we recall some concepts of complex dynamics that we use in this paper. These concepts can be completed in [14]. So, we need that nonlinear function f is defined on Riemann sphere $\hat{\mathbb{C}}$, as ∞ becomes one more point to be taken into account.

Let us assume that a fixed point iteration function acts on an arbitrary polynomial $p(z)$; that yields a rational function, that will be denoted by R. So, given any rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0,\ R(z_0),\ R^2(z_0), \ldots, R^n(z_0), \ldots\}.$$

Then, we analyze the phase plane of the map R by classifying the starting points from the asymptotical behavior of their orbits. A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_0) = z_0$ is satisfied. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$.

Moreover, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$. The fixed points different from those associated with the roots of the polynomial $p(z)$ are called *strange fixed points*.

A point z_0 is a *critical point* of the rational map R if R fails to be injective in any neighborhood of z_0. Indeed, if a critical point is different from those associated with the roots of the polynomial $p(z)$, it is called *free critical point*. Indeed, any superattracting fixed point is a critical point (let us remark that, if the iterative method has order of convergence at least two, the roots of $p(z)$ are superattracting fixed points).

The basin of attraction of an attractor α is defined as the set of points that, used as initial estimation, converge to α:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to \alpha,\ n \to \infty\}.$$

The Fatou set of the rational function R, $\mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point or periodic orbit). Its complement in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That means that the basin of attraction of any fixed or periodic point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The following classical result is a key fact to be used in the definition and interpretation of parameter planes. In it, the concept of immediate basin of attraction is introduced, that is, the connected component of the basin of attraction that includes the attracting fixed point.

Theorem 1 ([15,16]). *Let R be a rational function. The immediate basin of attraction of an attracting fixed or periodic point holds, at least, a critical point.*

The conjugacy classes are extremely useful because they allow us to get general results by using simple functions. Let f and g be functions defined and with image at Riemann sphere. An analytic conjugation between f and g is a diffeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $h \circ f = g \circ h$.

The following results assure us that, if our aim is to analyze the stability of Traub's method on cubic polynomials, it is enough to study its behavior on $p(z) = (z-1)(z-r)(z+1)$, as the dynamics are equivalent, that is, a conjugacy preserves fixed and periodic points as well as their character and basins of attraction.

Theorem 2 (Scaling Theorem [17]). *Let $f(z)$ be an analytic function, and let $T(z) = \alpha z + \gamma$, with $\alpha \neq 0$, be an affine map. If $g(z) = (f \circ A)(z)$, then $(T \circ R_g \circ T^{-1})(z) = R_f(z)$, that is, R_f is affine conjugated to R_g by T, where R_f and R_g denote the fixed point operator of Traub's method on f and g, respectively.*

Theorem 3 ([18]). *Let $q(z)$ be any cubic polynomial with simple roots. Then, it can be parametrized by means of an affine map to $p(z) = (z-1)(z-r)(z+1)$, $r \in \mathbb{C}$. This map induces a conjugacy between $R_q(z)$ and $R_p(z)$.*