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Schwarz type preconditioners for the neutron diffusion equation

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HIGHLIGHTS

- To solve the neutron diffusion equation many linear systems has to be solved using preconditioned Krylov methods.
- Traditional preconditioners based on incomplete factorizations are expensive in terms of memory.
- Domain decomposition preconditioners are studied including substructuring preconditioners and additive Schwarz preconditioners.
- 2D and 3D benchmarks have been studied obtaining better performance results than usual preconditioners.

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ABSTRACT

Domain decomposition is a mature methodology that has been used to accelerate the convergence of partial differential equations. Even if it was devised as a solver by itself, it is usually employed together with Krylov iterative methods improving its rate of convergence, and providing scalability with respect to the size of the problem.

In this work, a high order finite element discretization of the neutron diffusion equation is considered. In this problem the preconditioning of large and sparse linear systems arising from a source driven formulation becomes necessary due to the complexity of the problem. On the other hand, preconditioners based on an incomplete factorization are very expensive from the point of view of memory requirements. The acceleration of the neutron diffusion equation is thus studied here by using alternative preconditioners based on domain decomposition techniques inside Schur complement methodology. The study considers substructuring preconditioners, which do not involve overlapping, and additive Schwarz preconditioners, where some overlapping between the subdomains is taken into account.

The performance of the different approaches is studied numerically using two-dimensional and three-dimensional problems. It is shown that some of the proposed methodologies outperform incomplete LU factorization for preconditioning as long as the linear system to be solved is large enough, as it occurs for three-dimensional problems. They also outperform classical diagonal Jacobi preconditioners, as long as the number of

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systems to be solved is large enough in such a way that the overhead of building the preconditioner is less than the improvement in the convergence rate.

1. Introduction

Domain decomposition methods were first proposed by Schwarz [1] as an analytical tool. A renewed interest in this kind of methods was noticed with the appearance of parallel computers [2]. These methods were first proposed to solve partial differential equations on complex domains and later these techniques were extended to solve linear equations (see [3] and references therein). The algebraic Schwarz methods are based on partitioning the vector of unknowns into subsets, which correspond to a partition of the coefficients matrix, typically associated with different subdomains in the continuous problem. The solution of the whole system is achieved by solving the systems associated with the different blocks of the partitioned matrix, which are simpler problems than the original one. Schwarz methods to solve linear systems are not as competitive as other alternative methods such as multi-grid solvers, but they can be used as efficient preconditioners of Krylov methods, at the cost of a few more iterations.

Domain decomposition methods in nuclear engineering have been receiving increasing attention for the last years due to their potential to solve large problems by using a *divide-and-conquer* strategy. For example, a Schur complement is used to accelerate the core solver in [4]. In [5,6] a method based on the Schwarz iterative algorithm is studied to solve the mixed neutron diffusion and the simplified spherical harmonics neutron equation. Finally, the response matrix method that implements a two-level model, a global and a local level, was analysed in [7,8]. The local level is defined on a mesh fine enough to provide accurate results while the global level is defined on a coarse mesh which accelerates the convergence of the method. The methodology for linking the local and global solutions is the key aspect of the response matrix method.

The neutron diffusion equation is an approximation of the neutron transport equation [9]. This equation describes a balance between generation and loss of neutrons by a generalized differential eigenvalue problem. The dominant eigenvalue and its corresponding eigenfunction describe the steady state neutron distribution, thus, these quantities should be determined for most of the reactor analyses. The problem is discretized by a high order Galerkin Finite Element Method (FEM), thus transforming it into an algebraic eigenvalue problem. Different methods can be used to solve this eigenvalue problem, while the common bottle-neck of all of them is the solution of a large number of linear systems for a few different coefficient matrices.

Because of the discretization with a FEM, the matrices of the systems are large and sparse their related linear systems are symmetric and positive definite. Thus, these systems are well suited to be solved with an iterative Krylov subspace method, where a classical approach for preconditioning these linear systems is by using an incomplete factorization of the coefficient matrices [10]. Nevertheless, the computation of such preconditioner requires to store the coefficients matrices in the computer memory, in addition to the preconditioner itself, which results in large requirements of memory resources. These memory requirements can be lowered by different fill-in or threshold criteria for the preconditioner, although the minimum memory requirement remains large if a fast preconditioner is used. In this work, we study alternative preconditioning techniques for these systems based on the domain decomposition methodology, aiming at lowering the memory requirements when high order Finite Element Methods are used for the spatial discretization.

The rest of the paper is organized as follows. In Section 2, the high order finite element method (FEM) that discretizes the problem using Lagrange polynomials is briefly reviewed. These polynomials provide a partition of the shape functions set into vertices, edges, faces and interior functions. Using this natural partition, the linear systems of equations associated with each energy group can be solved with a Schur Complement method that algebraically decouples the interior degrees of freedom from the other ones. This method, also called static condensation method, is presented in Section 3. This method is advantageous when a high polynomial degree, p, is used in the FEM discretization. To precondition the resulting Schur complement system two different strategies are described in this work. First a substructuring block Jacobi preconditioner is studied in Section 4, where the coupling between the different elements is neglected. Also, a domain decomposition algorithm with overlapping between subdomains, like the additive Schwarz method, is considered in Section 5. Several benchmarks are studied in Section 6 to test numerically the performance of the different approaches proposed. Finally, the main conclusions of the paper are summarized in Section 7.

2. Neutron diffusion equation and its high order FEM discretization

The neutron diffusion equation is an approximation of the neutron transport equation relying on the assumption that the neutron current is proportional to the gradient of the neutron flux by means of a diffusion coefficient. This approximation is analogous to Fick's law in species diffusion and to Fourier's law in heat transfer [9]. For a given configuration of a nuclear reactor core, it is always possible to force its criticality dividing the neutron production rate by a positive number, λ,