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The weak Galerkin method for solving the incompressible Brinkman flow

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ABSTRACT

The Brinkman equations are used to describe the dynamics of fluid flows in complex porous media, with the high variability in the permeability coefficients, which may take extremely large or small values. This paper is devoted to the numerical analysis of a family of weak Galerkin (WG) finite element methods for solving the time-dependent Brinkman problems. This WG method is equipped with stable finite elements consisting of usual polynomials of degree $k \geq 1$ for the velocity and polynomials of degree $k-1$ for the pressure. The velocity element is enhanced by polynomials of degree k on the interface of the finite element partition. All the finite element functions are discontinuous for which the usual gradient and divergence operators are implemented as distributions in properly-defined spaces. We further establish a priori error estimates in L^2 norm and H^1 norm, and we provide a few numerical experiments to illustrate the behavior of the proposed scheme and confirm our theoretical findings regarding optimal convergence of the approximate solutions.

1. Introduction

The viscous flow of incompressible fluids in complex porous media was extensively investigated in the last several decades [1–4] due to its wide applications in industry and science, such as petroleum industry, underground water hydrology, mangrove swamps, biomedical engineering, and heat pipes modeling, etc. The common feature of these problems is that the media are composed by complex cavity. The permeability varies highly on several scales. The problem is governed by the so-called Brinkman equations [5] as follows:

$$\mathbf{u}_t - \mu\Delta\mathbf{u} + \nabla p + \mu\kappa^{-1}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \times [0,T], \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \times [0,T], \tag{1.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \times [0,T], \tag{1.3}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}^0 \quad \text{in } \Omega, \tag{1.4}$$

where μ is the fluid viscosity and κ denotes the permeability tensor of the porous media. $\mathbf{u}$ and p represent the velocity and pressure of the fluid, and $\mathbf{f}$ is a momentum source term. $\mathbf{u}_t$ is the time partial derivative of $\mathbf{u}(\mathbf{x},t)$. $\mathbf{f} \in [L^2(0,T;H^s(\Omega))]^d$, $\mathbf{g} \in [L^2(0,T;H^{s+1}(\partial\Omega))]^d$, and $\mathbf{u}^0 \in [H^{s+1}(\Omega)]^d$, where $s \geq 0$ is an integer. We assume that the system is defined in a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$, for $t \in [0,T]$. For simplicity, we consider (1.1) and (1.3) with $\mu = 1$ and $\mathbf{g} = \mathbf{0}$ (note that one can always scale the solution with μ).

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Assume that there exist two positive numbers $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \boldsymbol{\xi}^t \boldsymbol{\xi} \le \boldsymbol{\xi}^t \kappa^{-1} \boldsymbol{\xi} \le \lambda_2 \boldsymbol{\xi}^t \boldsymbol{\xi}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d.$$

Here $\boldsymbol{\xi}$ is understood as a column vector and $\boldsymbol{\xi}^t$ is the transpose of $\boldsymbol{\xi}$. We consider the case where λ_1 is of unit size and λ_2 is possibly of large size.

The above parabolic system has been treated by various numerical methods, for example, the finite element methods(FEMs) [6–10], the mixed FEMs [11–14], finite volume methods [15,16], the method of characteristics type [17,18], and discontinuous Galerkin methods [19–21]. The goal of this paper is to present a newly developed weak Galerkin finite element method for the time-dependent Brinkman equations which are based on the definitions of discrete weak divergence and gradient operators introduced in [22].

The weak Galerkin (WG) method was first introduced in 2012 [23] for the second order elliptic problem [24–27], biharmonic problem [28–31] and further developed for other applications, such as the Stokes [32,33], Maxwell [34], etc. Its central idea is to interpret the partial differential operators as generalized distributions over the space of discontinuous functions including boundary information, and employ some proper stabilizations to enforce weak continuities for approximating functions. The WG methods, by design, use discontinuous piecewise polynomials. The WG methods are highly flexible in element construction and mesh generation. It enforces the weak continuity by introducing weakly define derivatives and parameter free stabilizers. Formulations of the WG methods can be easily obtained from the variational forms of the corresponding PDE by simply replacing derivatives by weakly defined derivatives and adding a stabilizer. The WG methods have wide range of applications in applied problems arising from science and engineering. The basic principles and some recent developments of weak Galerkin finite element methods have been reviewed in [35].

In this paper, we present a stable numerical method for the time-dependent Brinkman equations using weak Galerkin finite element methods. The WG method presents a natural and straightforward framework for constructing stable numerical algorithms for the Brinkman problem. Actually, as reported in Ref. [22,36], the Brinkman equation resembles features of either Darcy flow or Stokes flow, depending on the parameter μ. The weak Galerkin method is robust for the parameter μ according to [22]. It has been shown that the WG methods are efficient and robust by allowing the use of discontinuous approximating functions. The proposed WG method is equipped with stable finite elements consisting of polynomials of degree $k \ge 1$ for the velocity and polynomials of degree $k-1$ for the pressure. The velocity element is enhanced by polynomials of degree k on the interface of the finite element partition. The backward Euler Weak Galerkin method is defined by replacing the time derivative by a backward difference quotient for the fully-discrete scheme. We further establish a priori error estimates in L^2 norm and H^1 norm, and provide a few numerical experiments to illustrate the efficiency of our scheme.

2. The weak Galerkin method

In this section we design a semi-discrete and a fully-discrete weak Galerkin finite element schemes for the Brinkman problems (1.1)–(1.4).

This paper uses the standard definition for the Sobolev space $H^s(D)$ and the associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$ for any $s \ge 0$. The space $H^0(D)$ coincides with $L^2(D)$. When $D = \Omega$, we shall drop the subscript D and s in the norm and inner product notation. When D is an edge/face, we also use $\langle\cdot,\cdot\rangle_D$ to represent the L^2 inner product.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in $\mathbb{R}^2$ or polyhedral in $\mathbb{R}^3$ satisfying a set of conditions [27] to be specified, and T be each element with ∂T as its boundary. Denote by $\mathcal{F}_h$ the set of all edges or faces in $\mathcal{T}_h$. For any element $T \in \mathcal{T}_h$, denote by h_T the diameter of T. Similarly, the diameter of e is given by h_e. We shall define the mesh size of partition $\mathcal{T}_h$ as

$$h = \max_{T \in \mathcal{T}_h} h_T.$$

For each $T \in \mathcal{T}_h$, let $P_r(T)$ and $P_r(\partial T)$ be the sets of polynomials on T and ∂T with degree no more than r, respectively. We introduce weak Galerkin finite element spaces for the velocity function $\mathbf{u}$ and the pressure function p, as follows:

$$V_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \{\mathbf{v}_0, \mathbf{v}_b\}|_T \in [P_k(T)]^d \times [P_k(e)]^d, e \in \partial T, \mathbf{v}_b = \mathbf{0} \text{ on } \partial\Omega\},$$
$$W_h = \{q \in L_0^2(\Omega) : q|_T \in P_{k-1}(T)\},$$

where $L_0^2(\Omega) = \{q \in L_0^2(\Omega); \int_\Omega q dT = 0\}$. By a weak Galerkin function $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$, we mean $\mathbf{v} = \mathbf{v}_0$ inside of the element T and $\mathbf{v} = \mathbf{v}_b$ on the boundary of the element T. We would like to emphasize that any function $\mathbf{v} \in V_h$ has a single value $\mathbf{v}_b$ on each edge $e \in \mathcal{F}_h$.

The parabolic problem (1.1)–(1.4) can be written in the following variational formulation [37,38] which presented the well-posed analysis of this problem: Find $\mathbf{u} \in L^2(0, T; [H_0^1(\Omega)]^d)$ and $p \in L^2(0, T; L_0^2(\Omega))$ satisfying

$$(\mathbf{u}_t, \mathbf{v}) + (\nabla\mathbf{u}, \nabla\mathbf{v}) + (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}),$$
$$(\nabla\cdot\mathbf{u}, q) = 0,$$
$$\mathbf{u}(\cdot, 0) = \mathbf{u}^0,$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$ for $t \in (0, T]$.