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Analysis of a family of HDG methods for second order elliptic problems[☆]

Binjie Li, Xiaoping Xie [*]

School of Mathematics, Sichuan University, Chengdu 610064, China

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ABSTRACT

In this paper, we analyze a family of hybridizable discontinuous Galerkin (HDG) methods for second order elliptic problems in two and three dimensions. The methods use piecewise polynomials of degree $k \geqslant 0$ for both the flux and numerical trace, and piecewise polynomials of degree $k+1$ for the potential. We establish error estimates for the numerical flux and potential under the minimal regularity condition. Moreover, we construct a local postprocessing for the flux, which produces a numerical flux with better conservation. Numerical experiments in two-space dimensions confirm our theoretical results.

1. Introduction

The pioneering works on hybrid (also called mixed-hybrid) finite element methods are due to Pian [1] and Fraejis de Veubeke [2] for the numerical solution of linear elasticity problems. Here the term "hybrid", as stated in [3], means "the constraints on displacement continuity and/or traction reciprocity at the inter-element boundaries are relaxed a priori" in the hybrid finite element model. One may refer to [4–14] and to [15–17] respectively for some developments of hybrid stress (also called assumed stress) methods and hybrid strain (also called enhanced assumed strain) methods based on generalized variational principles, such as Hellinger–Reissner principle and Hu–Washizu principle. In [18–21], stability and convergence were analyzed for several 4-node hybrid stress/strain quadrilateral/rectangular elements. We refer to [22–24] for the analysis of hybrid methods for 4th order elliptic problems, and to [25–28] for the analysis for second-order elliptic boundary-value problems. One may see [29,24,30,10] for more references therein on the hybrid methods.

Due to the relaxation of function continuity at the inter-element boundaries, the hybrid finite element model allows for piecewise-independent approximation to the displacement/potential or stress/flux solution, thus leading to a sparse, symmetric and positive definite discrete system through local elimination of unknowns defined in the interior of the elements. This is one main advantage of the hybrid methods. The process of local elimination is also called "static condensation" in engineering literature. In the discrete system, the unknowns are only the globally coupled degrees of freedom of the approximation trace of the "displacement" or "traction" defined only on the boundaries of the elements.

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* Corresponding author.
E-mail addresses: libinjiefem@yahoo.com (B. Li), xpxie@scu.edu.cn (X. Xie).

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In [31] Cockburn et al. introduced a unifying framework for hybridization of finite element methods for the second order elliptic problem: find the potential u and the flux $\boldsymbol{\sigma}$ such that

$$\begin{cases} \boldsymbol{c}\boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{\sigma} = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a polyhedral domain, $\boldsymbol{c}(x) \in [L^\infty(\Omega)]^{d\times d}$ is a matrix valued function that is symmetric and uniformly positive definite on $\Omega, f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Here hybridization denotes the process to rewrite a finite element method in a hybrid version. The unifying framework includes as particular cases hybridized versions of mixed methods [32–34], the continuous Galerkin (CG) method [35], and a wide class of hybridizable discontinuous Galerkin (HDG) methods. In [31] three new kinds of HDG methods, or more precisely LDG-H (Local DG-hybridizable) methods, were presented, where the unknowns are the approximations of the potential u and flux $\boldsymbol{\sigma}$, defined in the interior of elements, and the numerical trace of u, defined on the interface of the elements. In [36] an error analysis was carried out for one of the three HDG methods by [31], based on the use of a projection operator inspired by the form of the numerical traces of the methods. Following the same idea as in [36], a unifying framework was proposed in [37] to analyze a large class of methods including the hybridized versions of some mixed methods as well as several HDG methods. We note that in [38], a reduced HDG scheme was proposed, which only includes the potential approximation and numerical trace as unknowns. Recently this HDG method was analyzed in [39] for the Poisson problem.

In this paper, we analyze a family of HDG methods for problem (1.1). We use piecewise polynomials of degree k for both the numerical flux $\boldsymbol{\sigma}_h$, and the numerical trace λ_h of u, and use piecewise polynomials of degree $k+1$ for the numerical potential u_h. It should be mentioned that, in [40,41], the same methods have been analyzed for convection diffusion equations with constant diffusion coefficient and linear elasticity problems, respectively. We note that in our analysis the diffusion coefficient $\boldsymbol{c}(x) \in [L^\infty(\Omega)]^{d\times d}$ is a matrix valued function. By following a similar idea of [42], we establish error estimates for the numerical flux and potential under the minimal regularity condition, i.e.

$$u \in H^1(\Omega) \quad \text{and} \quad \boldsymbol{\sigma} \in H(\text{div}; \Omega).$$

This is significant since the regularity $u \in H^{1+\alpha}(\Omega)$ may not hold for $\alpha > 0.5$ for practical problems. To our best knowledge, such a kind of error estimation has not been established for HDG methods, yet. We note that in [43] an error estimate was established under the condition that $u \in H^{1+\alpha}(\Omega)$ $(\alpha > 0.5)$, while the estimate with $\alpha < 0.5$ is fundamental to the multi-grid method developed there. In our contribution, we also construct a local postprocessing for the flux, which produces a numerical flux $\boldsymbol{\sigma}_h^*$ with better conservation.

The rest of this paper is organized as follows. In Section 2 we follow the general framework in [31] to describe the corresponding HDG methods. Section 3 is devoted to the error estimation for the numerical flux and potential under the minimal regularity condition. Section 4 presents the postprocessing for the flux. Finally Section 5 provides numerical results.

2. HDG method

Let us start by introducing some geometric notations. Let $\mathcal{T}_h$ be a conventional conforming and shape-regular triangulation of Ω, and let $\mathcal{F}_h$ be the set of all faces of $\mathcal{T}_h$. For any $T \in \mathcal{T}_h$, we denote by h_T the diameter of T and set $h := \max_{T\in\mathcal{T}_h} h_T$. For any $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$, let $V(T)$, $M(F)$ and $\boldsymbol{W}(T)$ be local spaces of finite dimensions. Then we define

$$V_h := \left\{ v_h \in L^2(\Omega) : v_h|_T \in V(T) \text{ for all } T \in \mathcal{T}_h \right\}, \tag{2.1}$$

$$M_h := \left\{ \mu_h \in L^2(\mathcal{F}_h) : \mu_h|_F \in M(F) \text{ for all } F \in \mathcal{F}_h \right\}, \tag{2.2}$$

$$\boldsymbol{W}_h := \left\{ \boldsymbol{\tau}_h \in [L^2(\Omega)]^d : \boldsymbol{\tau}_h|_T \in \boldsymbol{W}(T) \text{ for all } T \in \mathcal{T}_h \right\}. \tag{2.3}$$

For any $g \in L^2(\partial\Omega)$, set

$$M_h(g) := \left\{ \mu_h \in M_h : \langle \mu_h, \eta_h \rangle_{\partial\Omega} = \langle g, \eta_h \rangle_{\partial\Omega} \text{ for all } \eta_h \in M_h \right\},$$

and define $M_h^0 := M_h(0)$. In addition, for any $T \in \mathcal{T}_h$, define

$$M(\partial T) := \left\{ \mu \in L^2(\partial T) : \mu|_F \in M(F) \text{ for all face } F \text{ of } T \right\}, \tag{2.4}$$

and then define $\mathcal{P}_T^\partial : H^1(T) \to M(\partial T)$ by

$$\left\langle \mathcal{P}_T^\partial v, \mu \right\rangle_{\partial T} = \langle v, \mu \rangle_{\partial T} \quad \text{for all } v \in H^1(T) \text{ and } \mu \in M(\partial T). \tag{2.5}$$

Above and in what follows, for any polyhedral domain $D \subset \mathbb{R}^d$, we use $(\cdot,\cdot)_D$ and $\langle \cdot,\cdot \rangle_{\partial D}$ to denote the L^2-inner products in $L^2(D)$ and $L^2(\partial D)$ respectively, and for convenience, we shall use $(\cdot,\cdot)$ to abbreviate $(\cdot,\cdot)_\Omega$.