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Immersed finite element methods for unbounded interface problems with periodic structures[☆]

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ABSTRACT

Interface problems arise in many physical and engineering simulations involving multiple materials. Periodic structures often appear in simulations with large or even unbounded domain, such as magnetostatic/electrostatic field simulations. Immersed finite element (IFE) methods are efficient tools to solve interface problems on a Cartesian mesh, which is desirable to many applications like particle-in-cell simulation of plasma physics. In this article, we develop an IFE method for an interface problem with periodic structure on an infinite domain. To cope with the periodic boundary condition, we modify the stiffness matrix of the IFE method. The new matrix is maintained symmetric positive definite, so that the linear system can be solved efficiently. Numerical examples are provided to demonstrate features of this method.

1. Introduction

In engineering simulations, many devices characterize highly regular extended periodic structures, such as optics system in ion thruster [1,2]. Numerical methods are usually applicable when such infinite samples possess a periodicity known *a priori*. In the mean time, only finite regions of these infinite structures are of practical interest. Simulation on a finite region is usually computationally feasible through existing numerical methods. One frequently used approach is to model only a small part of the infinite sample by truncating outer boundaries. This approach often assumes the normal derivative of the potential to be zero at the periodic boundaries. For example, the homogeneous Neumann boundary condition is enforced in the simulation of spacecraft–ion thruster interaction in [1]. However, the normal derivative of the potential is continuous at the periodic boundary edges but not necessary to be zero. For electromagnetic field problems, such as plasma–environment interaction simulation, periodic boundary conditions (PBC) are an important tool to understand the behavior of periodic structures. Many techniques have been developed in the study of PBC, and they have been used extensively in the simulation of periodical electromagnetic fields [3–6].

To simulate the behaviors of plasma medium, the particle-in-cell (PIC) method [7–10] models plasma as many macro-particles and follows the evolution of the orbits of individual particles in the self-consistent electromagnetic field. This

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method includes two main processes which are often called "scatter" (for finding the charge density from the particle positions) and "gather" (for interpolating the particle forces from the potential at the grid points). The PIC method is difficult to be utilized with body-fitting meshes, because locating millions of particles in an unstructured grid costs significant searching time in both "gather" and "scatter" steps in a typical PIC computational cycle; hence, it is preferable to use a more structured mesh to fulfil the process via indexing with the coordinates. Moreover, media materials in a PIC method are often separated by an interface with complicated geometry (see [9,11,12]). As a result, it is unrealistic to expect optimal convergent solution in a structured non-body-fitting mesh simply by conventional finite element methods.

Recently, a new type of field solvers, named immersed finite element (IFE) methods [13–20] have been developed to solve interface problems based on Cartesian meshes. In an IFE method, material interfaces are allowed to cut through some elements which are called interface elements, distinguished from those non-interface elements which are disjoint from interfaces. Standard finite element basis functions are adopted on non-interface elements, and piecewise polynomials are used to construct special IFE basis function on interface elements to mimic the behavior of the exact solution. For interface problems with nonhomogeneous flux jump conditions, a homogenization with the level set method was developed in [21]. Alternatively, the local IFE space can be enriched by adding a basis function that can capture the nonhomogeneous flux jump [22]. Recently, this idea was extended to solve a moving interface problem with nonhomogeneous flux jump [23]. We refer to [24] for IFE methods in handling nonhomogeneous jump for both solution and flux.

In this article, we will propose an IFE method for solving an unbounded interface problem with periodic structure. To simulate the behavior in a finite domain, we truncate the unbounded domain and impose appropriate PBC. We will modify the global stiffness matrix to accommodate the PBC. Such modification is only carried out in the periodic variation domain. The new global matrix is preserved symmetric positive definite, so that the linear system can be solved efficiently.

The rest of this article is organized as follows. In Section 2, we introduce the unbounded interface problem and develop an IFE method for the truncated problem with PBC. In Section 3, we present the matrix modification technique to accommodate PBC. Numerical examples and some related discussion are provided in Section 4. Finally, in Section 5 we draw some conclusions.

2. IFE methods for interface problems

In this section, we will first present the unbounded elliptic interface problem and the truncated model problem with periodic boundary condition in a finite domain. Next we will recall some IFE methods for elliptic interface problem with homogeneous [25] and nonhomogeneous flux jump [22]. To make our presentation concise, we only consider the bilinear IFE methods based on rectangular meshes in the article. For linear IFE methods on triangular meshes, the discussion is similar and we refer readers to [26,27] for more details.

2.1. An unbounded elliptic interface problem

Let $\hat{\Omega} = (-\infty, \infty) \times (0, 1)$ be an unbounded (in x-direction) domain. A smooth interface curve $\hat{\Gamma}$ separates $\hat{\Omega}$ into sub-domains $\hat{\Omega}^-$ and $\hat{\Omega}^+$. We consider the following second-order elliptic interface problem

$$-\nabla \cdot (\beta \nabla u) = f \quad \text{in } \hat{\Omega}^+ \cup \hat{\Omega}^-. \tag{2.1}$$

Assume that the solution u has the periodic structure in the x-direction, *i.e.*, there is a positive constant L such that

$$u(x, y) = u(x + L, y), \quad \forall x \in (-\infty, \infty). \tag{2.2}$$

In y-direction, we assume that the solution satisfies the Dirichlet boundary condition

$$u(x, 0) = g_0(x), \tag{2.3}$$

$$u(x, 1) = g_1(x). \tag{2.4}$$

Across the interface $\hat{\Gamma}$, the solution is assumed to satisfy the interface jump conditions

$$[\![u]\!]_{\hat{\Gamma}} = 0, \tag{2.5}$$

$$\left[\!\!\left[\beta \frac{\partial u}{\partial n}\right]\!\!\right]_{\hat{\Gamma}} = Q. \tag{2.6}$$

Here $[\![v]\!]_{\hat{\Gamma}} = (v|_{\Omega^+})_{\hat{\Gamma}} - (v|_{\Omega^-})_{\hat{\Gamma}}$ is the jump of v across $\hat{\Gamma}$. The function Q is the jump of normal flux, and n is the unit normal vector of $\hat{\Gamma}$. The flux jump is said to be homogeneous if $Q = 0$; otherwise it is nonhomogeneous. Also, we assume that the coefficient function β is periodic in the x-direction of periodicity L. Without loss of generality, the coefficient $\beta(x, y)$ is discontinuous across the interface $\hat{\Gamma}$. For simplicity, we assume that β is a piecewise constant function as follows:

$$\beta(x, y) = \begin{cases} \beta^-, & (x, y) \in \hat{\Omega}^-, \\ \beta^+, & (x, y) \in \hat{\Omega}^+. \end{cases}$$