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A discontinuous Galerkin least-squares finite element method for solving coupled singularly perturbed reaction–diffusion equations

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ABSTRACT

A discontinuous Galerkin least-squares finite element method is proposed to solve coupled reaction–diffusion equations with singular perturbations. This method produces solutions without numerical oscillations when uniform meshes are used. Numerical examples are provided to demonstrate the efficiency of the method.

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1. Introduction

Consider the following coupled system of M singularly perturbed reaction–diffusion equations

$$\mathcal{L}\mathbf{u} := -E^2\Delta\mathbf{u} + A\mathbf{u} = \mathbf{f} \quad \text{in } \Omega := (0,1)^d, \tag{1a}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{1b}$$

where $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \ldots, u_M(\mathbf{x}))^T$, $E = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_M)$ is an $M \times M$ diagonal matrix with parameters $0 < \varepsilon_i \le 1$, $A(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{M\times M}$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_M(\mathbf{x}))^T$ are sufficiently smooth, and $d = 1, 2$ or 3 is the dimension of region Ω. To ensure that (1) has a solution, it is assumed that there are constants α and β such that [1]

$$\alpha := \min_{1\le i\le M} \min_{\mathbf{x}\in\overline{\Omega}} a_{ii}(\mathbf{x}) > 0, \tag{2a}$$

$$a_{ij} \le 0 \quad \text{on } \overline{\Omega} \text{ for } i \ne j,\ 1 \le i, j \le M, \tag{2b}$$

and

$$0 \le \beta := \max_{1\le i\le M} \beta_i < 1, \tag{2c}$$

where

$$\beta_i = \max_{\mathbf{x}\in\overline{\Omega}} \frac{1}{a_{ii}(\mathbf{x})} \sum_{\substack{j=1\\ j\ne i}}^{M} |a_{ij}(\mathbf{x})|, \quad 1 \le i \le M.$$

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The focus of this paper is on numerical approximation of (1) with singular perturbations; i.e. when some or all parameters $\varepsilon_i \ll 1$. This problem is very challenging, as solutions of such problems typically possess boundary layers, a very small region in which solution values change severely. To resolve the layers, graded meshes, such as Bakhvalov meshes, Shishkin meshes, and other adaptive meshes, have been applied in finding numerical solutions of this type of problems. Nevertheless, when the parameters ε_i are in different orders of magnitudes, it is very difficult to design a working graded mesh (cf., e.g., [2]).

In this work, we will propose a discontinuous Galerkin (DG) least-squares (LS) finite element method (FEM) for singularly perturbed problem (1). The highlight of the method is its numerical stability: optimal solutions without numerical oscillations can be obtained on a uniform mesh of big size. This method is a natural combination of the LS method and the DG method, which has been introduced to solve single convection–reaction–diffusion equations with singular perturbation [3–5]. It has desirable properties of both the LS and the DG methods. In particular, the LS FEM is a mixed method minimizing the residuals in a least-squares sense, which has uniform formulation for different problems. For linear differential equations, it leads to symmetric positive-definite algebraic systems which can be efficiently solved by iterative methods. Comparing with general mixed methods, the LS FEM does not require the inf–sup conditions on the finite element spaces. It has been applied to solve a wide scope of problems. For more details on the theory and applications of LS FEMs, we refer to the books [6,7] and the references therein. The DG method, on the other hand, was initially introduced to solve neutron transport equations, which has been widely applied to solve different types of problems. Allowing for discontinuities in the trial and test spaces, the DG method offers an efficient approach to resolve interfaces and layers. For more information about the DG method, the readers are referred to the books [8,9] and the references therein.

In this paper, we will introduce the DG LS FEM for problem (1) and present numerical examples without providing rigorous analysis. In particular, in the rest of this section, notations will be introduced. In Section 2, the DG LS FEM will be introduced with details. In Section 3, several numerical examples will be reported to show the efficiency of the method. Concluding remarks will be given in Section 4.

1.1. Notations

Throughout this paper, we shall use C to denote a generic positive constant which is independent of ϵ_i, $1 \le i \le M$ and of the mesh used. Column vectors and scalars are denoted by bold and plain letters, respectively.

We will denote the inner products in $L^2(\Omega)$ and Cartesian powers of $L^2(\Omega)$ by $(\cdot,\cdot)_\Omega$. For $1 \le p \le \infty$ and $s \ge 0$, we use the standard notations for the Sobolev space $W_p^s(\Omega)$ and its norm $\|\cdot\|_{W_p^s(\Omega)}$ and seminorm $|\cdot|_{W_p^s(\Omega)}$. $H^s(\Omega)$ is used to stand for the space $W_2^s(\Omega)$, whose norm and seminorm are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, respectively. When no confusion may arise, the measure Ω will be omitted from the above norm designations. We recall the space $H_0^1(\Omega)$ consisting of all functions in $H^1(\Omega)$ that vanish on the boundary $\partial\Omega$, and the space

$$H(\mathrm{div};\Omega) = \left\{\mathbf{q} \in [L^2(\Omega)]^d : \nabla\cdot\mathbf{q} \in L^2(\Omega)\right\}.$$

We define further the vector function space

$$\mathbf{H}(\Omega) = H(\mathrm{div};\Omega) \times H_0^1(\Omega)$$

with the following energy norms in $\mathbf{H}(\Omega)$ for $1 \le i \le M$

$$\|(\mathbf{q};v)\|_{0,\varepsilon_i}^2 = \varepsilon_i^2\|\mathbf{q}\|_0^2 + \|v\|_0^2, \tag{3}$$

$$\|(\mathbf{q};v)\|_{1,\varepsilon_i}^2 = \varepsilon_i^4\|\nabla\cdot\mathbf{q}\|_0^2 + \varepsilon_i^2|v|_1^2 + \|(\mathbf{q};v)\|_{0,\varepsilon_i}^2. \tag{4}$$

Here, we denote the norms on Cartesian powers $[H^s(\Omega)]^d$ also by $\|\cdot\|_{s,\Omega}$, or simply $\|\cdot\|_s$, where there is no chance for ambiguity.

Let $\mathcal{T}_h = \{\Omega_k\}_{k=1}^N$ be a regular triangulation on Ω with mesh size $h = \max_{1\le k\le N}\mathrm{diam}(\Omega_k)$. Let $\mathcal{E}$ be the union of the boundaries of all elements Ω_k associated with the partition $\mathcal{T}_h$, and $\mathcal{E}_{int} \subset \mathcal{E}$ be the set of all interior edges in Ω. Note that graded meshes (e.g. Bakhvalov meshes and Shishkin meshes) can be used, which however are not necessary for the method proposed in below. In this paper, unless otherwise specified, our discussions are on uniform meshes.

We shall use the following broken Sobolev spaces

$$H^s(\mathcal{T}_h) = \left\{v \in L^2(\Omega) : v|_{\Omega_k} \in H^s(\Omega_k),\ 1 \le k \le N\right\},$$

for $s \ge 0$ and

$$\mathbf{H}(\mathcal{T}_h) = \left\{(\mathbf{q};v) \in H(\mathrm{div};\mathcal{T}_h) \times H^1(\mathcal{T}_h) : v|_{\partial\Omega} = 0\right\},$$

where

$$H(\mathrm{div};\mathcal{T}_h) = \left\{\mathbf{q} : \mathbf{q}|_{\Omega_k} \in [L^2(\Omega_k)]^d,\ \nabla\cdot\mathbf{q}|_{\Omega_k} \in L^2(\Omega_k),\ 1 \le k \le N\right\}.$$

The inner products and norms for the continuous spaces can be analogously defined for the broken Sobolev spaces. In particular, for $(\mathbf{q};v) \in \mathbf{H}(\mathcal{T}_h)$, we define its energy norms for $1 \le i \le M$

$$\|(\mathbf{q};v)\|_{0,\varepsilon_i}^2 = \sum_{k=1}^N \|(\mathbf{q};v)\|_{0,\varepsilon_i,\Omega_k}^2 \quad\text{and}\quad \|(\mathbf{q};v)\|_{1,\varepsilon_i}^2 = \sum_{k=1}^N \|(\mathbf{q};v)\|_{1,\varepsilon_i,\Omega_k}^2, \tag{5}$$