Journal of Computational and Applied Mathematics 307 (2016) 204–215

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Global error analysis and inertial manifold reduction[☆]

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ARTICLE INFO

Article history:
Received 1 September 2015
Received in revised form 13 February 2016

Keywords:
Global error
Lyapunov exponent
Inertial manifold

ABSTRACT

Four types of global error for initial value problems are considered in a common framework. They include classical forward error analysis and shadowing error analysis together with extensions of both to include rescaling of time. To determine the amplification of the local error that bounds the global error we present a linear analysis similar in spirit to condition number estimation for linear systems of equations. We combine these ideas with techniques for dimension reduction of differential equations via a boundary value formulation of numerical inertial manifold reduction. These global error concepts are exercised to illustrate their utility on the Lorenz equations and inertial manifold reductions of the Kuramoto–Sivashinsky equation.

1. Introduction

In many complex systems modeled using differential equations the slow dynamics drive the system. There is a vast literature on inertial manifold techniques to determine the mapping between the slow dynamics and the fast dynamics. This decouples the system and focuses attention on the (often) low dimensional slow dynamics that drive the system. Once such a low dimensional reduction is achieved, then one would like to infer the behavior of the system from simulations of the reduced equations. An often overlooked problem is in assessing whether the global error on the inertial manifold can be controlled and in what sense. The standard approach to global error analysis is classical forward error analysis for initial value differential equations in which the initial condition is the same for both the exact solution and the numerical approximation. Shadowing error analysis generalizes this in a significant way by allowing for slightly different initial conditions for the exact and approximate solutions. This expands the class of problems for which long time error statements are possible, from contractive problems to those with a splitting between expansive and contractive modes. A further refinement that has been investigated in the shadowing literature involves the rescaling of time when differential equations have a non-trivial attractor.

Our contribution in this paper is to develop a unified approach to global error analysis for initial value problems that can be used to determine when there is uncertainty in the numerical approximation of solutions of differential equations; we also show that the technique is applicable in the context of inertial manifold dimension reduction. We report on initial numerical experiments for a new inertial manifold reduction technique combined with an assessment of the global error in approximating the reduced equations. The inertial manifold technique which we outline here and describe in more detail

☆ The research was support in part by NSF grants DMS-1115408 and DMS-1419047 and University of Kansas Undergraduate Research Awards.

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http://dx.doi.org/10.1016/j.cam.2016.02.023

in [1] involves first performing a time dependent linear decoupling transformation and then determining the mapping between the slow and fast dynamics implicitly by solving a boundary value problem. Information obtained during the solution of the boundary value problem is then employed to assess the relationship between the local error and if possible the global error as characterized by forward or shadowing error analysis and with or without rescaling of time. In this paper we highlight four different types of long time global error analysis. We will also exercise recently developed techniques for dimension reduction in differential equations in the context of these types of global error analysis.

Shadowing based techniques for global error analysis involve relaxing the requirement that the initial conditions for the exact solution and the numerical solution agree. This has the effect that the linearized error equation need not be solved forward in time. This allows for positive global error statements for a larger class of problems over long time intervals, i.e., for problems that are not contractive such as in the case of a system with positive Lyapunov exponents. Shadowing also provides a framework to allow for rescaling of time, i.e., allowing for perturbations in the time step, (see the work of Coomes, Kocak, and Palmer [2,3], and Van Vleck [4,5]). Rescaling of time is especially important when there is a periodic orbit or more general non-trivial attractor [6]. Work on numerical shadowing includes the ground breaking work of Hammel, Yorke, and Grebogi [7,8], the work of Chow, Lin, and Palmer (e.g. [9,10]), the numerical work of Sauer and Yorke [11], and the initial work on breakdown of shadowing of Dawson, Grebogi, Sauer, and Yorke [12].

Inertial manifolds, first introduced by Foias, Sell, and Temam [13] for dissipative dynamical systems, are finite dimensional, exponentially attracting, positively invariant Lipschitz manifolds. Similar concepts are slow manifolds in slow–fast system introduced in meteorology and widely used in weather forecasting [14–17], and center-unstable manifolds in the classic sense. In fact, [18] shows that a slow manifold is a special type of inertial manifold, and as mentioned in the original work [13], it can be described as a global center-unstable manifold. The main application is the inertial manifold reduction, meaning the restriction of the dynamical system to the inertial manifold where the long-term dynamics coincide with those of the original system without introducing errors. In particular, since the manifold is finite dimensional, the reduced system is also finite dimensional, whereas the original system may arise from an infinite dimensional system. Because of its importance, there has been tremendous work in regard to its theory and computation, see e.g. [19–25] and [19–24,26,25], respectively. Recently, the theory of inertial manifolds has been generalized to non-autonomous dynamical systems [27–30], and recently, to random dynamical systems [31], and [32] (and the references therein).

We take the approach here of decoupling the time-dependent linear part of the equation using techniques that have proven useful in the approximation of Lyapunov exponents. We first employ an orthogonal change of variables $Q(t)$ that brings the time dependent coefficient matrix for the linear part of the equation to upper triangular. Subsequently, we will compute a change of variables that decouples the linear part. This then gives us equations of the form considered by Aulbach and Wanner in [33]. A similar change of variables has been employed to justify that Lyapunov exponents and Sacker–Sell spectrum may be obtained from the diagonal of an upper triangular coefficient matrix (see section 5 of [34] and sections 4 and 5 of [35]). The Refs. [36,37] (see also the references therein) provide a summary and overview of recent work on approximation of Lyapunov exponents and in obtaining the orthogonal change of variables $Q(t)$.

This paper is outlined as follows. We first present a framework for global error analysis in Section 2. Techniques to be employed for non-autonomous inertial manifold reduction are in Section 3. In Section 4 we outline of methods to determine the amplification of the local error that determines the global error. This is followed by details of our dimension reduction implementation based upon time dependent linear decoupling transformation and subsequent solution of the inertial manifold equations using a boundary value differential equation solver. Section 5 contains the results of the technique applied to the three dimensional Lorenz 1963 model and to an inertial manifold reduction of a Galerkin approximation of the Kuramoto–Sivashinsky equation.

2. Framework for global error analysis

In this section we present a framework for global error analysis of initial value differential equations. We will focus our attention on four specific characterizations of global error analysis. The differences among the characterizations is determined by which variables are allowed to differ between the numerically computed solution and an exact solution. This follows the framework developed for shadowing based error analysis in [4].

To make these ideas concrete consider a smooth initial value ODE of the form

$$\dot{u} = f(u, t), \qquad u(t_0) = u_0. \tag{2.1}$$

If we let $\varphi(u_n, h_n; t_n)$ denote the solution operator that advances the state variable u_n, h_n time units from t_n, then the exact solution satisfies (for $t_{n+1} = t_n + h_n$),

$$u(t_{n+1}) = \varphi(u(t_n), h_n; t_n).$$

A general approach to global error analysis can be obtained using the setup employed in numerical shadowing. Outlined below are four measures of the computational error in approximating the solution to an initial value differential equations. Subsequently, we will apply these ideas to the reduction obtained on the inertial manifold to assess to the computational error in approximating solutions to these reduced set of equations.

The idea behind shadowing type global error analysis is to use a numerical approximation of the solution as an initial guess for a functional Newton-type iteration and show that this converges to a nearby exact solution. If we let $x = \{x_n\}_{n=0}^{N}$