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Numerical methods for Riemann–Hilbert problems in multiply connected circle domains

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ABSTRACT

Riemann–Hilbert problems in multiply connected domains arise in a number of applications, such as the computation of conformal maps. As an example here, we consider a linear problem for computing the conformal map from the exterior of m disks to the exterior of m linear slits with prescribed inclinations. The map can be represented as a sum of Laurent series centered at the disks and satisfying a certain boundary condition. R. Wegmann developed a method of successive conjugation for finding the Laurent coefficients. We compare this method to two methods using least squares solutions to the problem. The resulting linear system has an underlying structure of the form of the identity plus a low rank operator and can be solved efficiently by conjugate gradient-like methods.

1. Introduction

Linear Riemann–Hilbert (R–H) problems arise in a number of applications, such as the computation of conformal maps [1–3]. For multiply connected domains in the complex plane, maps from domains bounded by circles are useful for computations, since Laurent series can be used to represent the functions analytic in the exterior of the disks. Here we compare numerical methods for a simple R–H problem for computing the conformal map from a domain exterior to circles to a domain exterior to a number of linear slits. The methods solve for the *Laurent coefficients* and include a method of *successive conjugation* due to Wegmann [4,3] and a method based on a *least squares solution* to the boundary value problem; see, e.g., [5,6]. We formulate the least squares problem in such a way that the resulting system has singular values well-grouped around 1 with an underlying structure of the *identity plus a low-rank matrix*. Conjugate-gradient-like methods can therefore often be used efficiently. The main point of this paper is to uncover this structure in a simple example and investigate its effect on the numerics. We expect that the analysis here will be useful for a number of other similar computational problems, such as those in [7–12].

In Section 2, we introduce the conformal map from the exterior of m given disks to the exterior of m slits with given inclinations and show that it satisfies a Riemann–Hilbert boundary value problem for a function analytic in the circular domain. Section 3 reviews Wegmann's method of successive conjugation for solving this Riemann–Hilbert problem and presents our least squares method. An analysis of the linear systems shows the grouping of the eigenvalues and their effect on the convergence of the conjugate gradient method applied to the normal equations. Section 4 gives several numerical examples showing the behavior of the methods for domains of various connectivity and cases where the circles nearly touch. Section 5 discusses possible future work. Portions of the MATLAB code are given in the Appendix.

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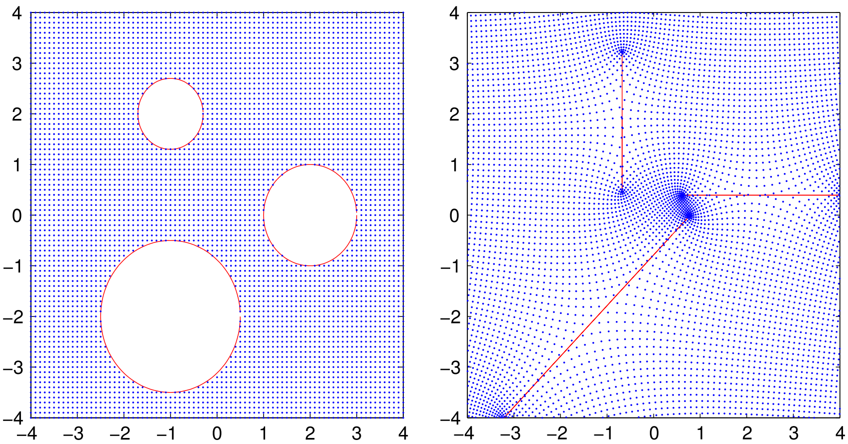

Fig. 1. Map to $m = 3$ slits from [3] with circle centers, $z_1 = 2, z_2 = -1-2i, z_3 = -1+2i$ and radii, $R_1 = 1, R_2 = 1.5, R_3 = 0.7$ and inclination angles of slits, $\alpha_1 = 0, \alpha_2 = \pi/4, \alpha_3 = \pi/2$.

2. Conformal mapping

Let G denote the domain exterior to m mutually exterior, nonoverlapping disks in the complex plane bounded by circles with centers, z_k, and radii, R_k, $k = 1, \ldots, m$. Since the circles do not overlap, $(R_j + R_k)/|z_j - z_k| < 1$. Wegmann [3, Eq. (390)] considers the conformal map,

$$\Phi(z) = z + i \cdot \Psi(z) \tag{1}$$

from G to an assembly of linear slits inclined at angles α_k. Here $\Psi(z)$ is analytic in G and $\Psi(\infty) = 0$, so Ψ can be represented as a sum of m Laurent series centered at the z_k's, (or, more precisely, the Taylor expansions centered at ∞ and converging in the exterior of the disks.) Given the circles and the normalization $z + O(1/z)$ at ∞ the map $\Phi(z)$ is uniquely determined by standard theorems; see [13, Thm 17.6a]. The example from [3] in Fig. 1 illustrates the map. Since $\Phi(z)$ can be continued analytically by reflection across the circles into the interior of the disks, as in, e.g., [9], the series converge geometrically.

We will use the notation in [3] for ease of comparison.

The m circles C_k, $k = 1, \ldots, m$ are parametrized by $z_{|k} := z_k + R_k e^{-it}$, $t \in [0, 2\pi]$ with the domain G to the left. In general, we denote values of functions on the kth circle by, e.g., $\Phi_{|k} := \Phi(z_{|k}) = z_{|k} + i\Psi_{|k}$. Since Φ maps the kth circle to a slit of inclination α_k, it must satisfy

$$\mathrm{Im}\left[e^{-i\alpha_k} \cdot \Phi_{|k}\right] = A_k, \text{ constant.}$$

Wegmann converts this into a boundary condition for $\Psi(z)$ as follows. Using $\Phi_{|k} = z_{|k} + i\Psi_{|k}$, we have that

$$\begin{aligned}\mathrm{Im}\left[e^{-i\alpha_k} \cdot \Phi_{|k}\right] &= \mathrm{Im}\left[e^{-i\alpha_k} \cdot \left(z_k + R_k e^{-it} + i \cdot \Psi_{|k}\right)\right] \\ &= \mathrm{Im}\left[e^{-i\alpha_k} z_k\right] + R_k \mathrm{Im}\left[e^{-i(\alpha_k + t)}\right] + \mathrm{Im}\left[ie^{-i\alpha_k}\Psi_{|k}\right] \\ &= \mathrm{Im}\left[e^{-i\alpha_k} z_k\right] - R_k \sin(t + \alpha_k) + \mathrm{Re}\left[e^{-i\alpha_k}\Psi_{|k}\right] \\ &= A_k\end{aligned}$$

giving the boundary conditions,

$$\mathrm{Re}\left[e^{-i\alpha_k} \cdot \Psi_{|k}(t) + a_{k0}\right] = R_k \sin(t + \alpha_k) =: \psi_k(t), \tag{2}$$

where $t \in [0, 2\pi]$, $k = 1, \ldots, m$, and $a_{k0} := \mathrm{Im}\left[e^{-i\alpha_k} z_k\right] - A_k$ are m real unknowns.

2.1. Riemann–Hilbert (R–H) problems

Wegmann [3] states the following theorem.

Theorem 1. *For any integer $l \geq 0$ and for any sufficiently smooth functions ψ_k on the boundary of G the R–H problem*

$$\mathrm{Re}\left(e^{i\lambda_k} e^{ilt} \Psi_{|k} + a_{kl} e^{ilt} + \cdots + a_{k1} e^{it} + a_{k0}\right) = \psi_k,$$

has a unique solution consisting of an analytic function Ψ in G, with $\Psi(\infty) = 0$, and complex numbers $a_{k1}, \ldots, a_{kl}$ and real a_{k0}.

Here we just consider the case $l = 0$ and we have $\lambda_k = -\alpha_k$, the inclination angles of the slits. Additional theoretical discussion and applications can also be found in, e.g. [12], and references therein.