Journal of Computational and Applied Mathematics 307 (2016) 284–306

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Stable and convergent approximation of two-dimensional vector fields on unstructured meshes

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ARTICLE INFO

Article history:
Received 30 July 2015
Received in revised form 26 January 2016

MSC:
65N06
65N08
65N12
65N22
76D07

Keywords:
Staggered-grid
MAC
Incompressible Stokes
Finite difference
Finite volume
Unstructured meshes

ABSTRACT

A new framework is proposed for analyzing staggered-grid finite difference finite volume methods on unstructured meshes. The new framework employs the concept of external approximation of function spaces, and gauge convergence of numerical schemes through the quantities of vorticity and divergence, instead of individual derivatives of the velocity components. The construction of a stable and convergent external approximation of a simple but relevant vector-valued function space is demonstrated, and the new framework is applied to establish the convergence of the MAC scheme for the incompressible Stokes problem on unstructured meshes.

1. Introduction

In simulations of physical systems, it is often advantageous to stagger the vectorial variables with the scalar variables. The resulting schemes are collectively called staggered-grid schemes. A classical example of staggered-grid scheme is the Marker-and-Cell scheme [1], also known as the C-grid in the geoscience community [2], in which the mass and other related variables are specified at cell centers and the normal velocity components are specified at cell edges. The MAC scheme is widely accepted as the method of choice for incompressible flows; see [3] for a review. Since its introduction, it has also been argued that the scheme is suitable for flows at all speeds; see the seminal papers by [4,5], as well as later developments by [6–10]. For geophysical flows, the C-grid scheme has been shown to be superior in resolving inertial-gravity dispersive relations; see the seminal paper by [2], and recent expositions on this topic, [11–14]. In recent years, to take advantage of the growing power of supercomputers, there has been a push to extend the C-grid scheme onto unstructured meshes for complex problems on complex geometric domains. In this regard, we mention the work by [15–17].

In this work we concern ourselves with the analysis of staggered-grid numerical schemes on unstructured meshes. Staggered-grid schemes are mostly constructed using the finite difference (FD) or finite volume (FV) techniques, and therefore, with regard to analysis, they pose the same challenges that classical FD/FV schemes do, namely the lack of variational formulations and the use of low-order piecewise constant functions. Staggered-grid schemes on unstructured meshes pose an extra challenge: the normal and/or tangential velocity components specified on the edges may not align with the canonical directions of the original vector field. The last two decades have seen quite some efforts on this topic;

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http://dx.doi.org/10.1016/j.cam.2016.01.049

the theory for the MAC-scheme on structured meshes is fairly complete, at least when classical fluid problems, such as compressible/incompressible Stokes, are concerned. In 1975, roughly 10 years after the MAC scheme was introduced, Girault [18] proposed a finite element method on "interlaced" rectangular meshes for the stationary incompressible Navier–Stokes problem; the method reduces to the classical MAC scheme when the boundaries of the domain align with the mesh lines and a special 4-point quadrature rule is used. First-order error estimates were given for both the velocity and the pressure. Through the co-volume approach, Nicolaides and Wu [19] derives the a priori error estimates of the MAC scheme on rectangular meshes for the stationary two-dimensional incompressible Navier–Stokes problem. Kanschat [20] shows that, on rectangular meshes, the MAC scheme for incompressible flows is algebraically equivalent to the divergence-conforming discontinuous Galerkin method based on the lowest order Raviart–Thomas elements. Eymard et al. [21] perform the stability and convergence analysis of the MAC scheme for the two- and three-dimensional compressible Stokes problem on rectangular meshes. Chénier et al. [22] consider a variational extension of the MAC to the full Navier–Stokes equations on semi-regular, non-conforming, and locally refined meshes. E and Liu [23] analyze the MAC scheme for the two-dimensional time-dependent Navier–Stokes equations, again, on rectangular meshes. The situation on unstructured meshes is quite different. The only work known to us on this topic is Nicolaides [24], who derives the a priori error estimates for the MAC scheme for the incompressible Stokes problem on unstructured meshes. We should point out that Chou [25] derives the a priori error estimates for MAC-like schemes for generalized Stokes on triangular meshes. But the schemes are constructed by approximating both of the canonical velocity components with piecewise linear Petrov–Galerkin elements, and thus are different from the type of schemes considered here.

We aim to develop a new theoretical framework for analyzing staggered-grid schemes for a wide range of fluid problems. There are two essential ingredients to this new framework. The first is the concept of external approximation of function spaces, which was proposed by Céa [26], and extensively used by Aubin [27] and Temam [28]. This concept has been used recently by several authors to study the convergence of non-staggered finite volume schemes [29,30]. Formal definitions will be given in the next section. Briefly speaking, external approximation adds an auxiliary function space F alongside the original function space V and the discrete function space V_h (see Fig. 1). With the aid of several mappings defined between these function spaces, elements from V_h, which are often discontinuous, can now be compared with elements from V in the auxiliary space F. The second ingredient of our new framework is the use of vorticity and divergence to gauge the convergence of the numerical schemes. This is a direct reflection of the fact that staggered-grid schemes are best at mimicking the vorticity and/or divergence, but not the canonical components of the velocity field, or any of its gradients in the canonical directions.

The framework is general enough to be applicable to different types of staggered-grid schemes (MAC, co-volume, etc.), and potentially to a wide range of fluid problems (compressible/incompressible Stokes, shallow water equations, etc.) The goal of the current article is to present the analysis framework and to apply it to the first case of interest, the classical incompressible Stokes problem. The existence and uniqueness of a discrete solution, and its convergence to the true solution are established.

After we have completed this work, we were made ware that very similar results have been obtained in [31]. But the current work and the cited work differ in the approaches taken. We utilize an external approximation framework for vector fields for the convergence analysis, while [31] rely on a strong reconstruction operator (to reconstruct the velocity field from either the tangential or normal velocity components) and a compactness result. They also prove the convergence for the pressure field for one version of the MAC schemes, which comes as an extra bonus of their approach. The same issue is not discussed in our work, because the pressure field disappear in the variational form for the problem. On the other hand, it appears that the results of [31] only apply to triangular-Delaunay meshes, while ours apply to arbitrarily unstructured staggered grids. Due to these differences, we are comfortable in publishing this work.

The rest of the article is arranged as follows. In Section 2, we recall the definitions of external approximations, and present the framework for constructing and analyzing external approximations of vector fields on unstructured meshes. In Section 3, we apply the framework to analyze the MAC scheme for the incompressible Stokes problem. We finish in Section 4 with some concluding remarks concerning the current work and future plans.

2. Approximation of vector fields

In the study of partial differential equations (PDEs) governing physical systems, such as fluids, various vector-valued function spaces may appear as the natural setting of the problems. These function spaces usually differ in the level of the regularity and boundary behaviors. We let Ω be a bounded and simply connected domain on the two-dimensional plane with piecewise smooth boundaries, and in this section, we consider the vector-valued function space

$$V = H_0^{\mathrm{div}}(\Omega) \cap H^{\mathrm{curl}}(\Omega). \tag{2.1}$$

Here $H_0^{\mathrm{div}}(\Omega)$ is a space of square-integrable vector-valued functions whose divergence is also square-integrable, and whose normal component vanishes on the boundary (see [32] for details). Similarly, $H^{\mathrm{curl}}(\Omega)$ denotes a space of square-integrable vector-valued functions whose curl is also square-integrable. By [32, Proposition 3.1], the space V is algebraically and topologically included in the space $H^1(\Omega)$, and in addition, the H^1-norm of functions from V can be majorized by the