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Improved optimal conditions and iterative parameters for the optimal control problems with an integral constraint in square ☆

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ABSTRACT

A distributed optimal control problem is considered with an inequality constraint on the state variable. And the constraint reads that the integral of the state is not more than a given positive constant. An efficient approach is introduced to investigate optimality conditions of this problem. Based on the Uzawa algorithm, an efficient algorithm is designed and its convergence is discussed with details. Especially, the bounds of two iterative parameters are investigated. With the Legendre–Galerkin spectral method, numerical results show that the algorithm is highly feasible.

1. Introduction

In applications, many engineering simulations are reformulated by optimal control problems (OCPs, for short). Over the past decades, there have been extensive researches on theories of OCPs. The control-constrained OCPs are well studied in [1–5] and the references cited therein. Meanwhile the state-constrained OCPs are investigated in [6–10] and among others. In numerous scientific and engineering applications, one may focus on some integral constraints on the state variable, such as not less than or not more than a positive constant. In this paper, we concern with an efficient approach to study the optimality condition of a distributed optimal control model. Especially, an improved formula of the multiplier is stated with a denominator. Based on the Uzawa algorithm, we design an efficient algorithm and study the iteration parameters in details.

The rest of this paper is organized as follows. The next section is devoted to proposing and analyzing the model problem. Specially, we investigate the corresponding improved optimality condition of the model problem and construct its Legendre–Galerkin spectral approximations. In Section 3, an efficient algorithm is designed with the Uzawa algorithm to solve the constrained OCPs. Meanwhile the bounds of iteration parameters are analyzed in detail. In Section 4, a numerical example is presented to illustrate our theoretical results. Finally, the conclusions and future work are listed in Section 5.

2. Optimal control problem and its spectral approximations

Let $\Omega \subset \mathbb{R}^2$ be a bounded open and convex set. $L^2(\Omega)$ and $H^m(\Omega)$ denote the Sobolev spaces with norms $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{H^m(\Omega)} = \|\cdot\|_{m,\Omega}$, respectively [11]. $U = L^2(\Omega)$ and $V = H_0^1(\Omega)$ are the control space and state space,

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respectively. In this paper, we choose $\Omega = (-1, 1)^2$ for simplicity. The problem we are interested in is the following distributed convex optimal control problem with an integral constraint on the state variable

$$\min_{y\in K} J(y) = \|y - y_d\|^2_{0,\Omega} + \|u\|^2_{0,\Omega} \tag{2.1}$$

subjects to

$$\begin{cases} -\Delta y = f + u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

And the constraint reads

$$K = \left\{ v \in V : \int_\Omega v \le \gamma \right\}, \tag{2.3}$$

where $\gamma > 0$, u is the control, y is the state, f and y_d are given smooth functions. $J(y)$ in (2.1) leads to the built-in convexity and coercivity, i.e., for $\forall v, w \in U$, there exist positive constants α, β $(\alpha \le \beta)$ such that

$$\begin{aligned} &(J'(v) - J'(w), v - w) \ge \alpha \|v - w\|^2_{0,\Omega}, \\ &\|J'(v) - J'(w)\|_{0,\Omega} \le \beta \|v - w\|_{0,\Omega}, \end{aligned} \tag{2.4}$$

which are key points for the existence and uniqueness of an optimal solution of (2.1)–(2.3). To derive an equivalent weak form of the model problem, for $\forall x, v \in U$, $\forall w, z \in V$, we define

$$(x, v) = \int_\Omega xv, \qquad a(w, z) = (\nabla w, \nabla z).$$

Then the weak form of the optimal control problem (2.1)–(2.3) reads

$$(\mathscr{P}) \begin{cases} \min\limits_{y\in K} \; J(y) = \|y - y_d\|^2_{0,\Omega} + \|u\|^2_{0,\Omega}, \\ \text{s.t. } a(y, w) = (f + u, w), \quad \forall w \in V. \end{cases} \tag{2.5}$$

Existence and uniqueness of the solution for $(\mathscr{P})$ can be obtained by standard techniques, and more details please refer to [5,9,12,13].

Lemma 2.1 *([14]). If u is the solution of an inequality constrained optimal control problem, for $\forall v \in U$, there exist corresponding constants t_i $(i = 1, 2, \ldots, m)$ such that*

$$t_i \ge 0, \qquad t_i F_i(u) = 0, \qquad \frac{\partial L(u, t_i)}{\partial u} \cdot v = 0, \tag{2.6}$$

where the Lagrange functional $L(u, t_i) : U \times \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$L(u, t_i) = J(y(u)) + \sum_{i=1}^{m} t_i F_i(y(u)), \qquad F_i(u) = \int_\Omega y(u) - \gamma_i, \tag{2.7}$$

and the convex functional $J(y(u))$ is stated in (2.1).

Theorem 2.1. *The pair $(y, u) \in V \times U$ is the solution of $(\mathscr{P})$ if and only if there exists a unique pair $(y^*, t) \in V \times \{\mathbb{R}^+ \cup 0\}$ such that*

$$(\mathscr{Q}) \begin{cases} (a) \quad a(y, w) = (f + u, w), & \forall\, w \in V, \\ (b) \quad a(y^*, w) = \left(y - y_d + \dfrac{t}{2}, w \right), & \forall\, w \in V, \\ (c) \quad (t, v - y) \le 0, & \forall\, v \in K, \\ (d) \quad y^* + u = 0. \end{cases} \tag{2.8}$$

Proof. In this proof, we just sketch the main technique. Obviously, by the first Green formula, we deduce from (2.2) that (2.8)-(a) holds. Set $m = 1$ in Lemma 2.1. For $\forall v \in K$, there holds

$$(t, v - y(u)) = t\int_\Omega v - t\int_\Omega y(u) = \begin{cases} t\left(\displaystyle\int_\Omega v - \gamma\right) \le 0, & \text{if } \displaystyle\int_\Omega y(u) = \gamma, \\ 0, & \text{if } \displaystyle\int_\Omega y(u) < \gamma. \end{cases}$$