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Projection algorithms for solving nonmonotone equilibrium problems in Hilbert space

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ABSTRACT

We propose two projection algorithms for solving an equilibrium problem where the bifunction is not required to be satisfied any monotone property. Under assumptions on the continuity, convexity of the bifunction and the nonemptyness of the solution set of the Minty equilibrium problem, we show that the sequences generated by the proposed algorithms converge weakly and strongly to a solution of the primal equilibrium problem respectively.

1. Introduction

Let $\mathbb{H}$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Let Ω be an open convex subset in $\mathbb{H}$ containing a nonempty closed convex C, and $f : \Omega \times \Omega \to \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for every $x \in C$. We consider the following equilibrium problem (shortly EP(C, f)) in the sense of Blum, Muu and Oettli [1,2], which is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C,$$

and its associated equilibrium problem.

$$\text{Find } u \in C \text{ such that } f(y, u) \leq 0, \quad \forall y \in C. \tag{1.1}$$

We call problem (1.1) as the Minty equilibrium problem (MEP(C, f) for short) due to M. Castellani and M. Giuli [3]. By S and S_M, we denote the solution set of EP(C, f) and MEP(C, f) respectively. While we denote by '$\to$' the strong convergence and by '$\rightharpoonup$' the weak convergence in the Hilbert space $\mathbb{H}$.

Although problem EP(C, f) has a simple formulation, it includes, as special cases, many important problems in applied mathematics: variational inequality problem, optimization problem, fixed point problem, saddle point problem, Nash equilibrium problem in noncooperative game, and others; see, for example, [4,1,2,5], and the references quoted therein.

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Solution methods for equilibrium problem have been usually extended from those for variational inequality problem [6–9] and other related problems; see, for instance, [10–17]. Among them, the extragradient method which was introduced by Korpelevich [18] for solving variational inequality problem and recently extended to equilibrium problem [19,20] is an important method. However, in our best knowledge, to implement this method, it always requires the solution set S of EP(C, f) is contained in the solution set S_M of MEP(C, f). This condition is guaranteed under the pseudomonotonicity assumption of bifunction f on C, that is, if $x, y \in C$, $f(x, y) \geq 0$, then $f(y, x) \leq 0$. Therefore, if S is not contained in S_M, then the existing extragradient method cannot be applied for EP(C, f) directly. For instance, take $C = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and for each $x = (x_1, x_2), y = (y_1, y_2) \in C$, we define bifunction f by the following formula

$$f(x, y) = |x_1 + x_2|(y_1 - x_1 + y_2^2 - x_2^2),$$

it is clear that $S = \{(-1, 0), (t, -t) : t \in [-1, 1]\}$; $S_M = \{(-1, 0)\}$, and $S \not\subset S_M$.

Note that, for each $y \in C$, by setting

$$L(y) = \{u \in C : f(u, y) \geq 0\}, \qquad L_M(y) = \{v \in C : f(y, v) \leq 0\},$$

then we can verify that

$$S = \cap_{y \in C} L(y), \qquad S_M = \cap_{y \in C} L_M(y).$$

If $f(\cdot, y)$ is upper semicontinuous on C for each $y \in C$, then $L(y)$ is closed for every $y \in C$, hence S is a closed set. While if $f(x, \cdot)$ is lower semicontinuous and quasiconvex on C for each $x \in C$, then $L_M(y)$ is closed and convex for all $y \in C$. Consequently, S_M is a closed and convex set and the Minty equilibrium problem reduces to a so-called convex feasibility problem [21]. If, in addition, the constraint set C is compact or f satisfies some certain coercive conditions [13] then S is nonempty [22,23]. However, S is not necessary convex as the above example.

Remember that, if $f(\cdot, y)$ is upper semicontinuous on C for each $y \in C$ and $f(x, \cdot)$ is convex for every $x \in C$, then we have $S_M \subset S$; see, for example [24].

In this paper, we propose two projection algorithms for solving the equilibrium problem in a real Hilbert space without pseudomonotonicity assumption of the bifunction, we assume that S_M is nonempty instead. The first algorithm can be considered as an extension of the one introduced by M. Ye and Y. He [25] for solving nonmonotone variational inequality problem in the Euclidean space, and the second one is a combination between the projection algorithm for solving the pseudomonotone equilibrium problem in the finite dimensional space in [26] and hybrid cutting technique proposed by W. Takahashi et al. [27] (see also Y. Censor et al. [28]).

The paper is organized as follows. The next section contains some preliminaries on the metric projection and equilibrium problems. Section 3 is devoted to presentation of a projection algorithm for EP(C, f) and its weak convergence. A strong convergence algorithm for EP(C, f) is presented in Section 4. The last section, is devoted to present an application of the proposed algorithm for Nash–Cournot equilibrium models of electricity markets and its implementation.

2. Preliminaries

In the rest of this paper, by P_C we denote the metric projection operator on C, that is

$$P_C(x) \in C : \ \|x - P_C(x)\| \leq \|y - x\|, \quad \forall y \in C,$$

and $d(., C)$ stands for the distance function to C, i.e.,

$$d(x, C) = \inf\{\|x - y\| : y \in C\}.$$

For example, if $H = \{y \in \mathbb{H} : \langle w, y - y^0\rangle \leq 0\}$ for some $w, y^0 \in \mathbb{H}$, then

$$d(x, H) = \begin{cases} \dfrac{|\langle w, x - y^0\rangle|}{\|w\|} & \text{if } x \notin H \\ 0 & \text{if } x \in H. \end{cases}$$

The following well known results on the projection operator onto a closed convex set will be used in the sequel.

Lemma 2.1. *Suppose that C is a nonempty closed convex subset in $\mathbb{H}$. Then*

(a) *$P_C(x)$ is singleton and well defined for every x;*
(b) *$z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in C$;*
(c) *$\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \ \forall x, y \in C$.*

Definition 2.1. A bifunction $\varphi : C \times C \to \mathbb{R}$ is said to be jointly weakly continuous on $C \times C$ if for all $x, y \in C$ and $\{x^k\}$, $\{y^k\}$ are two sequences in C converging weakly to x and y respectively, then $\varphi(x^k, y^k)$ converges to $\varphi(x, y)$.

In the sequel, we need the following blanket assumptions

(A1) $f(x, .)$ is convex on Ω for every $x \in C$;