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A weak Galerkin finite element scheme for solving the stationary Stokes equations

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ABSTRACT

A weak Galerkin (WG) finite element method for solving the stationary Stokes equations in two- or three- dimensional spaces by using discontinuous piecewise polynomials is developed and analyzed. The variational form we considered is based on two gradient operators which is different from the usual gradient-divergence operators. The WG method is highly flexible by allowing the use of discontinuous functions on arbitrary polygons or polyhedra with certain shape regularity. Optimal-order error estimates are established for the corresponding WG finite element solutions in various norms. Numerical results are presented to illustrate the theoretical analysis of the new WG finite element scheme for Stokes problems.

1. Introduction

The aim of this paper is to present a novel weak Galerkin finite element method for solving the stationary Stokes equations. Let Ω be a polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$. As a model for the flow of an incompressible viscous fluid confined in Ω, we consider the following equations

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{g}, \quad \text{on } \partial\Omega, \tag{1.3}$$

for unknown velocity function $\mathbf{u}$ and pressure function p (we require that p has zero average in order to guarantee the uniqueness of the pressure). Bold symbols are used to denote vector- or tensor-valued functions or spaces of such functions. Here $\mathbf{f}$ is a body source term, $\mu > 0$ is the kinematic viscosity and $\mathbf{g}$ is a boundary condition that satisfies the compatibility

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condition

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}\, ds = 0,$$

where $\mathbf{n}$ is the unit outward normal vector on the domain boundary $\partial\Omega$.

This problem mainly arises from approximations of low-Reynolds-number flows. The finite element methods for Stokes and Navier–Stokes problems enforce the divergence-free property in finite element spaces, which satisfy the inf–sup (LBB) condition, in order for them to be numerically stable [1–5]. The Stokes problem has been studied with various different new numerical methods: [6–10].

Throughout this paper, we would follow the standard definitions for Lebesgue and Sobolev spaces: $L^2(\Omega), H^1(\Omega)$, $[L^2(\Omega)]^d$,

$$[H_0^1(\Omega)]^d = \{\mathbf{v} \in [H^1(\Omega)]^d : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}$$

and

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}$$

are the natural spaces for the weak form of the Stokes problem [3,11]. Denote $(\cdot, \cdot)$ for inner products in the corresponding spaces.

Next we assume that $\mu = 1$ and $\mathbf{g} = \mathbf{0}$. Then one of the variational formulations for the Stokes problem (1.1)–(1.3) is to find $\mathbf{u} \in [H_0^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$ such that

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla \cdot \mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \tag{1.4}$$

$$(\nabla \cdot \mathbf{u}, q) = 0, \tag{1.5}$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$. Here $\nabla\mathbf{u}$ denotes the velocity gradient tensor $(\nabla\mathbf{u})_{ij} = \partial_j \mathbf{u}_i$. It is well known that under our assumptions on the domain and the data, problem (1.4)–(1.5) has a unique solution $(\mathbf{u}; p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$.

For any $p \in L_0^2(\Omega)$, define a functional ∇p such that

$$\langle \nabla p, \mathbf{v} \rangle = -(\nabla \cdot \mathbf{v}, p), \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^d.$$

It is easy to know that the weak form (1.4)–(1.5) is also equivalent to the following variational problem: find $(\mathbf{u}; p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ such that

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) + \langle \nabla p, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v}), \tag{1.6}$$

$$\langle \nabla q, \mathbf{u} \rangle = 0, \tag{1.7}$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$. The unique solvability of (1.6)–(1.7) follows directly from that of (1.4)–(1.5).

The WG method refers to a general finite element technique for partial differential equations where differential operators are approximated as distributions for generalized functions. This method was first proposed in [12–14] for second order elliptic problem, then extended to other partial differential equations [15–20]. Weak functions and weak derivatives can be approximated by polynomials with various degrees. The WG method uses weak functions and their weak derivatives which are defined as distributions. The most prominent features of it are:

- The usual derivatives are replaced by distributions or discrete approximations of distributions.
- The approximating functions are discontinuous. The flexibility of discontinuous functions gives WG methods many advantages, such as high order of accuracy, high parallelizability, localizability, and easy handling of complicated geometries.

The above features motivate the use of WG methods for the Stokes equations. It can easily handle meshes with hanging nodes, elements of general shapes with certain shape regularity and ideally suited for hp-adaptivity. In [21], Wang et al. considered WG methods for the Stokes equations (1.4)–(1.5). Similarly, in [18], they presented WG methods for the Brinkman equations, which is a model with a high-contrast parameter dependent combination of the Darcy and Stokes models. The numerical method of [18] is based on the traditional gradient-divergence variational form for the Brinkman equations. In [22], we presented a new WG scheme based on the gradient–gradient variational form. It is shown that this scheme is suit for the mixed formulation of Darcy which would present a better approximation for this case. In fact, for complex porous media with interface conditions, people often use Brinkman–Stokes interface model to describe this problem, which is an ongoing work for us now. In order to present a more efficient WG scheme, we prefer to utilize this gradient–gradient weak form to approximate the model. In order to unify the weak form of this interface problem, we need the numerical analysis results of this form for Stokes problem. However, to the best of our knowledge, the numerical analysis of methods based on the variational form (1.6)–(1.7) has never been done before. Therefore in this paper, we propose a WG method based on the weak form (1.6)–(1.7) of the primary problem. In addition, if we choose high order polynomials to approximate the model and use Schur complement to reduce the interior DOF of the velocity and pressure by the boundary DOF, the total DOF of this new method is less than the scheme of [21].