Journal of Computational and Applied Mathematics 302 (2016) 186–199

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Further results on generalized multiple fractional part integrals for complex values

Zhongfeng Sun, Aijuan Li, Huizeng Qin *

School of Science, Shandong University of Technology, Zibo, Shandong 255049, PR China

A R T I C L E I N F O

Article history:
Received 15 July 2015
Received in revised form 31 October 2015

Keywords:
Multiple integrals
Fractional part
Riemann zeta function
Beta function
Incomplete Beta function

A B S T R A C T

In this paper, the following multiple fractional part integrals

$$I_{n,\beta}^{\alpha_1,\alpha_2,\cdots,\alpha_n}=\int_{[0,1]^n}\prod_{j=1}^{n}x_j^{\alpha_j}\left\{S_n^{-1}\right\}^{\beta}dx_1\cdots dx_n$$

and

$$J_{n,\beta}^{\alpha}=\int_{[0,1]^n}S_n^{\alpha}\left\{S_n^{-1}\right\}^{\beta}dx_1\cdots dx_n$$

are studied for positive integer n and complex values of $\alpha,\beta,\alpha_j(j=1,2,\cdots,n)$, where $\{u\}$ denotes the fractional part of u, $\Re(s)$ denotes the real part of s and $S_n=x_1+x_2+\cdots+x_n$. It is proved that $I_{1,\beta}^{\alpha}$ can be represented as a linear combination of the Riemann zeta function, the Beta function and Euler's constant as $\Re(\beta)>-1$. Moreover, $I_{n,\beta}^{\alpha_1,\alpha_2,\cdots,\alpha_n}$ can be expressed by $I_{n-1,\beta}^{\alpha_1,\alpha_2,\cdots,\alpha_{n-1}}$, the Beta function and the incomplete Beta function for $n=2,3$. In addition, the recurrence formula of $J_{n,\beta}^{\alpha}(n=2,3,\cdots)$ is established and $J_{n,\beta}^{\alpha}$ can be expressed by $I_{1,\beta}^{\alpha}$, logarithmic function and some binomial coefficients.

1. Introduction

In this paper, we introduce the notation $\{u\}$ to denote the fractional part of u which is defined by $\{u\}=u-[u]$, where $[u]$ denotes the floor function of u. In [1], Furdui expressed the fractional part integrals

$$\int_0^1\left\{\frac{1}{x}\right\}^n\left\{\frac{1}{1-x}\right\}^n dx,\quad n=1,2,\ldots \tag{1.1}$$

by Euler's constant or Riemann zeta functions. Moreover, Furdui also summarized some explicit expressions of the special class of single and double fractional part integrals in [2] and discussed the closed form of the following multiple fractional part integral in [3]

$$\int_{[0,1]^n}\left\{\frac{x_1\cdots x_{n-1}}{x_n}\right\}^k dx_1\cdots dx_n, \tag{1.2}$$

* Corresponding author.
E-mail addresses: zfsun@sdut.edu.cn (Z. Sun), juanzi612@163.com (A. Li), qin_hz@163.com (H. Qin).

http://dx.doi.org/10.1016/j.cam.2016.02.014

where $n \geq 3$ and $k \geq 1$ are integers. In [4], Yu proved that the multiple fractional part integral

$$\int_{[0,1]^n} \left\{ \frac{1}{x_1 \cdots x_n} \right\} dx_1 \cdots dx_n, \quad n = 1, 2, \ldots \tag{1.3}$$

can be represented as the linear combination of the Euler's constant and Stieltjes constants. It is worth noting that the following multiple fractional part integrals [5–9]

$$I_{n,m}^{p_1,p_2,\ldots,p_n} = \int_{[0,1]^n} \prod_{j=1}^{n} x_j^{p_j} \left\{ S_n^{-1} \right\}^m dx_1 \cdots dx_n \tag{1.4}$$

and

$$J_{n,m}^{p} = \int_{[0,1]^n} S_n^p \left\{ S_n^{-1} \right\}^m dx_1 \cdots dx_n \tag{1.5}$$

are considered for non-negative integers p, m, p_j $(j = 1, \ldots, n)$ and positive integer n, where $S_n = x_1 + x_2 + \cdots + x_n$. Qin calculated the integral $J_{2,1}^0$ which was proposed by Furdui as a problem in [5] and expressed the integrals $J_{n,m}^0$ $(n = 1, 2, 3)$ in terms of the Riemann zeta function and Euler's constant in [6]. In [7], Furdui shown that integrals $I_{1,m}^p$ and $J_{2,m}^0$ can be expressed by the Riemann zeta function and Euler's constant. Moreover, a recurrence formula for calculating the integral $J_{n,1}^p$ was given in [8]. In [9], Li proved that the integrals $I_{n,m}^{p_1,p_2,\ldots,p_n}$ $(n = 2, 3)$ and $J_{n,m}^p$ can be expressed as linear combinations of the Riemann zeta function, logarithmic function and some binomial coefficients. Recently, fractional integrals have been used to deal with physical problems, such as fractional differential equations [10,11], the multi-term time fractional diffusion–wave equations [12], fractional Sturm–Liouville boundary value problems [13].

The integrals (1.4) and (1.5) can be generalized to the following multiple integrals of fractional part

$$I_{n,\beta}^{\alpha_1,\alpha_2,\ldots,\alpha_n} = \int_{[0,1]^n} \prod_{j=1}^{n} x_j^{\alpha_j} \left\{ S_n^{-1} \right\}^\beta dx_1 \cdots dx_n \tag{1.6}$$

and

$$J_{n,\beta}^{\alpha} = \int_{[0,1]^n} S_n^\alpha \left\{ S_n^{-1} \right\}^\beta dx_1 \cdots dx_n, \tag{1.7}$$

where α, β, α_j $(j = 1, 2, \ldots, n) \in \mathbb{C}$ and $\Re(s)$ denotes the real part of the complex number s. For convenience, we introduce the following symbol

$$I_\beta^\alpha := I_{1,\beta}^\alpha = J_{1,\beta}^\alpha = \int_0^1 x^\alpha \left\{ \frac{1}{x} \right\}^\beta dx. \tag{1.8}$$

The aim of this paper is to calculate the integral I_β^α and derive the recurrence formula for $I_{n,\beta}^{\alpha_1,\alpha_2,\ldots,\alpha_n}$ $(n = 2, 3)$ and $J_{n,\beta}^\alpha$ $(n = 2, 3, \ldots)$.

The structure of this paper is as follows. In Section 2, we focus on the relation between I_β^α and the Riemann zeta function, the Beta function, Euler's constant. In Section 3, the recurrence formulas for $I_{n,\beta}^{\alpha_1,\alpha_2,\ldots,\alpha_n}$ $(n = 2, 3)$ are established. In Section 4, we concern about the recurrence formula for $J_{n,\beta}^\alpha$ $(n = 2, 3, \ldots)$ and the relation between $J_{n,\beta}^\alpha$ and I_β^α. A final conclusion is given in Section 5.

2. The evaluation of I_β^α

In this paper, the Beta function $B(\alpha_1, \alpha_2)$ [14] and the incomplete Beta function $B(x; \alpha_1, \alpha_2)$ [15] are defined by

$$B(\alpha_1, \alpha_2) = \int_0^1 t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1} dt, \qquad B(x; \alpha_1, \alpha_2) = \int_0^x t^{\alpha_1 - 1}(1-t)^{\alpha_2 - 1} dt, \quad 0 < x < 1, \tag{2.1}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\Re(\alpha_1), \Re(\alpha_2) > 0$. Moreover, there are some properties of $B(\alpha_1, \alpha_2)$ and $B(x; \alpha_1, \alpha_2)$ [14,15] expressed as follows

$$B(\alpha_1, \alpha_2) = B(x; \alpha_1, \alpha_2) + B(1 - x; \alpha_2, \alpha_1) \tag{2.2}$$

and

$$B(\alpha_2, \alpha_1) = B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}, \tag{2.3}$$