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Minimal Geršgorin tensor eigenvalue inclusion set and its approximation

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ABSTRACT

For a tensor, its minimal Geršgorin tensor eigenvalue inclusion set is presented. By establishing a sufficient and necessary condition for the elements belonging to this set, we give a sequence of approximation sets and prove that the limit of this sequence is the minimal Geršgorin tensor eigenvalue inclusion set for an irreducible tensor.

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1. Introduction

Let $N = \{1, 2, \dots, n\}$. For a complex (real) order m dimension n tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ (written $\mathcal{A} \in \mathbb{C}^{[m,n]}$ ($\mathbb{R}^{[m,n]}$), respectively), where

$$a_{i_1 \cdots i_m} \in \mathbb{C}\ (\mathbb{R}), \quad i_j = 1, \dots, n,\ j = 1, \dots, m.$$

We call a complex number λ as an eigenvalue of $\mathcal{A}$ and a nonzero complex vector x as an eigenvector of $\mathcal{A}$ associated with λ, if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1}$$

where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m \in N} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively. Note that there are other definitions of eigenvalue and eigenvectors, such as, H-eigenvalue, D-eigenvalue and Z-eigenvalue; see [1–3]. Obviously, the definition of eigenvalue for matrices follows from the case $m = 2$.

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Tensor eigenvalues and eigenvectors have received much attention recently in the literature [4–8,1,9–11]. Many important results on the eigenvalue problem of matrices have been successfully extended to higher order tensors; see [4,12,13,8,1,9,2,14,15]. In [1], Qi generalized Geršgorin eigenvalue inclusion theorem from matrices to real symmetric tensors, which can be easily extended to generic tensors; see [14].

Theorem 1 *([1]). Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *and* $\sigma(\mathcal{A})$ *be the spectrum of* $\mathcal{A}$*, that is,*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

Then

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}), \tag{2}$$

where

$$\Gamma_i(\mathcal{A}) = \left\{ z \in \mathbb{C} : |z - a_{i\cdots i}| \le r_i(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m \in N,\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\cdots i_m}| \right\}$$

and

$$\delta_{i_1 i_2\cdots i_m} = \begin{cases} \delta_{i_1 i_2\cdots i_m} = 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to see that when $m = 2$, Theorem 1 reduces to the well-known Geršgorin eigenvalue inclusion theorem of matrices [16,17]. Here, we call $\Gamma_i(\mathcal{A})$ the ith Geršgorin tensor eigenvalue inclusion set. Note that $\Gamma_i(\mathcal{A})$ is a closed set in the complex plane $\mathbb{C}$, Hence, $\Gamma(\mathcal{A})$, which consists of the n sets $\Gamma_i(\mathcal{A})$, is also closed and bounded in $\mathbb{C}$.

Also in [1], Qi obtain other interesting result on the Geršgorin tensor eigenvalue inclusion set $\Gamma(\mathcal{A})$.

Theorem 2 *([1]). If* $\Gamma_i(\mathcal{A})$ *is disjoint with the other* $\Gamma_j(\mathcal{A})$, $j \neq i$*, then there are exactly* $(m-1)^{n-1}$ *eigenvalues which lie in* $\Gamma_i(\mathcal{A})$*. Furthermore, if all of* $\Gamma_i(\mathcal{A}), i = l_1, l_2, \ldots, l_k$ *are connected but disjoint with the other* $\Gamma_j(\mathcal{A}), j \neq i$*, then there are exactly* $k(m-1)^{n-1}$ *eigenvalues which lie in* $\bigcup_{i=l_1,l_2,\ldots,l_k} \Gamma_i(\mathcal{A})$.

In this paper, we also focus on the Geršgorin tensor eigenvalue inclusion set, present a minimal Geršgorin tensor eigenvalue inclusion set, and give a sufficient and necessary condition for the elements belonging to this set. For an irreducible tensor, we give a sequence of approximation sets which approximates to its minimal Geršgorin tensor eigenvalue inclusion set.

2. Minimal Geršgorin tensor eigenvalue inclusion set

In this section, we present a minimal Geršgorin tensor eigenvalue inclusion set of a tensor and study its characteristic. First, a lemma is given.

Lemma 3 *([14]). Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *and* $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ *be a diagonal nonsingular matrix. If*

$$\mathcal{B} = (b_{i_1\cdots i_m}) = \mathcal{A} D^{-(m-1)} \overbrace{DD\cdots D}^{m-1},$$

where

$$b_{i_1\cdots i_m} = d_{i_1}^{-(m-1)} a_{i_1 i_2\cdots i_m} d_{i_2} \cdots d_{i_m}, \quad i_1, \ldots, i_m \in N,$$

then $\mathcal{A}$, $\mathcal{B}$ *have the same eigenvalues.*

From Lemma 3, we obtain the following tensor eigenvalue inclusion set.

Theorem 4. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *and* $x = (x_1, x_2, \ldots, x_n)^T$ *be an entrywise positive vector, i.e.,* $x = (x_1, x_2, \ldots, x_n)^T > 0$*. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma^x(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i^x(\mathcal{A}), \tag{3}$$

where

$$\Gamma_i^x(\mathcal{A}) = \left\{ z \in \mathbb{C} : |z - a_{i\cdots i}| \le r_i^x(\mathcal{A}) = \sum_{\substack{i_2,\ldots,i_m \in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{|a_{ii_2\cdots i_m}| x_{i_2}\cdots x_{i_m}}{x_i^{m-1}} \right\}.$$

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