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SIMPLE-like preconditioners for saddle point problems from the steady Navier–Stokes equations[☆]

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ABSTRACT

The semi implicit method for pressure linked equations (SIMPLE) preconditioner is further generalized to a class of SIMPLE-like (SL) preconditioners for solving saddle point problems from the steady Navier–Stokes equations. The SL preconditioners can be also viewed as a generalization of the relaxed deteriorated PSS (RDPSS) preconditioner proposed by Cao et al. (2015). Convergence analysis of the corresponding SL iteration is presented and the optimal iteration parameter is obtained by minimizing the spectral radius of the SL iteration matrix. Moreover, Krylov subspace acceleration of the SL preconditioning is studied. The SL preconditioned saddle point matrix is analyzed. Results about eigenvalue and eigenvector distributions and the minimal polynomial are derived. Numerical experiments from "leaky" two dimensional lid-driven cavity problems are given, which show the advantages of the SL preconditioned GMRES over the DPSS and RDPSS preconditioned ones for solving saddle point problems.

1. Introduction

Consider the incompressible Navier–Stokes equations of the form:

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) is an open bounded domain with boundary $\partial\Omega$. Here, $\nu > 0$ is the viscosity parameter, Δ is the vector Laplacian, ∇ is the gradient, $\nabla\cdot\mathbf{u}$ is the divergence of $\mathbf{u}$, $\mathbf{f}$ is a given external force field and $\mathbf{g}$ is the Dirichlet boundary data. The goal is to find the unknown velocity and pressure fields $\mathbf{u}$ and p. The convective term $\mathbf{u}\cdot\nabla\mathbf{u}$ makes this system nonlinear. Linearization of the Navier–Stokes system (1.1) by Picard fixed-point iteration results in a sequence of Oseen problems of the form

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{w}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the divergence free field $\mathbf{w}$ is the velocity field obtained from the previous Picard iteration step.

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Spatial discretizations of the Oseen equations (1.2) using the Ladyzhenskaya–Babuša–Brezzi (LBB) stable finite element methods [1] result in the saddle point problems of the following structure:

$$\mathcal{A}_+ X = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv c_+, \tag{1.3}$$

where now u and p represent discrete velocity and pressure, respectively. A is the discretization of the diffusion and convection terms, B^T is the discrete gradient, B is the discrete divergence, and f and g contain forcing and boundary terms. In general, $A \in \mathbb{R}^{n\times n}$ is a positive real matrix, i.e., its symmetric part is positive definite, $B \in \mathbb{R}^{m\times n}$ $(m < n)$ is a matrix of full rank, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$. Negativing the second block row of (1.3) equivalently yields the nonsymmetric saddle point problems

$$\mathcal{A} X = \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \equiv c. \tag{1.4}$$

The nonsymmetric formulation (1.4) is especially natural when A is nonsymmetric, but positive real. In fact, $\mathcal{A}$ is positive stable, i.e., the eigenvalues of $\mathcal{A}$ have positive real parts, see [2]. This can be advantageous when using certain Krylov subspace methods, like GMRES.

A variety of scientific computing and engineering applications can derive the augmented linear system (1.3) or (1.4), for example, computational fluid dynamics, mixed finite element approximation of elliptic PDEs, weighted and equality constrained least squares estimation. See [3–5] and the references therein for a broad overview of applications and numerical solution techniques of saddle point problems.

Due to the large and sparse structure of its coefficient matrix, iteration methods are more attractive for solving the saddle point problem (1.3) or (1.4). A large amount of efficient methods have been presented in the literature, such as Uzawa-type methods [6–10], Hermitian and skew-Hermitian splitting methods [11–14], SIMPLE-type methods [15,5,16–18] and so forth. Meanwhile, Krylov subspace methods [19,20] are considered more efficient in general. However, they are not competitive without good preconditioners to speed up their convergence. Important and efficient preconditioners include block and approximate Schur complement preconditioners [21–24], constraint preconditioners [25–27], augmented Lagrangian preconditioners [28–32], HSS preconditioners [2,33–37] and so on.

In [36], based on the factorized form of the deteriorated PSS (DPSS) preconditioner [35]:

$$\mathcal{P}_{\mathrm{DPSS}} = \frac{1}{2\alpha} \begin{pmatrix} \alpha I + A & 0 \\ 0 & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I & B^T \\ -B & \alpha I \end{pmatrix} \tag{1.5}$$

for the nonsymmetric saddle point problem (1.4), Cao et al. proposed the relaxed DPSS (RDPSS) preconditioner structured as

$$\mathcal{P}_{\mathrm{RDPSS}} = \frac{1}{\alpha} \begin{pmatrix} A & 0 \\ 0 & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I & B^T \\ -B & 0 \end{pmatrix} = \begin{pmatrix} A & \frac{1}{\alpha} A B^T \\ -B & 0 \end{pmatrix}, \tag{1.6}$$

which is much closer to the saddle point matrix $\mathcal{A}$ compared with $\mathcal{P}_{\mathrm{DPSS}}$. Here, $\alpha > 0$ and I denotes identity matrix of proper size. Convergence of the corresponding RDPSS iteration is analyzed and the optimal parameter which minimizes the spectral radius of the iteration matrix is derived in [36]. Besides, some spectrum properties of the RDPSS preconditioned matrix $\mathcal{P}_{\mathrm{RDPSS}}^{-1}\mathcal{A}$ are also described there.

In [18], the SIMPLE preconditioner for saddle point problem (1.3) was presented with the following structure

$$\mathcal{P}_{\mathrm{SIMPLE}} = \begin{pmatrix} A & AD^{-1}B^T \\ B & 0 \end{pmatrix}, \tag{1.7}$$

with $D = \mathrm{diag}(A)$ being the diagonal part of A. Eigenvalue analysis of the preconditioned matrix $\mathcal{P}_{\mathrm{SIMPLE}}^{-1}\mathcal{A}_+$ has been presented in [18]. Some eigenvalue bounds and the estimation for the spectral condition number are also given. We see that the RDPSS preconditioner (1.6) is somewhat similar to the SIMPLE preconditioner (1.7).

In this paper, a new generalized variant of the RDPSS preconditioner (1.6), which resembles the SIMPLE preconditioner (1.7), is presented for the nonsymmetric saddle point problem (1.4). We call it SIMPLE-like (SL) preconditioner, which is given by:

$$\mathcal{P}_{\mathrm{SL}} = \begin{pmatrix} A & \frac{1}{\alpha} A Q^{-1} B^T \\ -B & 0 \end{pmatrix}, \tag{1.8}$$

with $\alpha > 0$ and Q being an approximation of A. It is obvious that this preconditioner reduces to the RDPSS preconditioner when $Q = I$. Convergence properties of the corresponding SL iteration are analyzed and the optimal iteration parameter is obtained. Moreover, spectral properties of the SL preconditioned matrix $\mathcal{P}_{\mathrm{SL}}^{-1}\mathcal{A}$ are studied. Compared with the RDPSS