Journal of Computational and Applied Mathematics 302 (2016) 234–257

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Optimal error estimates for semidiscrete Galerkin approximations to equations of motion described by Kelvin–Voigt viscoelastic fluid flow model

Ambit K. Pany [a,b], Saumya Bajpai [c], Amiya K. Pani [d,*]

[a] *Department of Mathematics, Gandhi Institute for Technological Advancement, Bhubaneswar-752054, India*
[b] *Research Scholar, Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India*
[c] *Institute of Infrastructure Technology Research and Management, Ahmedabad-380026, India*
[d] *Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai-400076, India*

A R T I C L E I N F O

Article history:
Received 7 December 2014
Received in revised form 19 August 2015

Keywords:
Kelvin–Voigt viscoelastic model
a priori bounds
Global attractor
Semidiscrete Galerkin approximation
Optimal error estimates
Uniqueness condition

A B S T R A C T

In this paper, a finite element Galerkin method is applied to equations of motion arising in the Kelvin–Voigt viscoelastic fluid flow model, when the forcing function is in $L^\infty(\mathbf{L}^2)$. Some *a priori* estimates for the exact solution, which are valid uniformly in time as $t \mapsto \infty$ and even uniformly in the retardation time κ an $\kappa \mapsto 0$ are derived. It is shown that the semidiscrete method admits a global attractor. Further, with the help of *a priori* bounds and Sobolev–Stokes projection, optimal error estimates for the velocity in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$-norms and for the pressure in $L^\infty(L^2)$-norm are established. Since the constants involved in error estimates have an exponential growth in time, therefore, in the last part of the article, under certain uniqueness condition, the error bounds are established which are valid uniformly in time. Finally, some numerical experiments are conducted which confirm our theoretical findings.

1. Introduction

Consider the following system of partial differential equations arising in the Kelvin–Voigt model of viscoelastic fluid flow:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \kappa \Delta \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(x, t), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

and incompressibility condition

$$\nabla \cdot \mathbf{u} = 0, \quad x \in \Omega,\ t > 0, \tag{1.2}$$

with initial and boundary conditions

$$\mathbf{u}(x, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0, \quad \text{on } \partial\Omega,\ t \geq 0, \tag{1.3}$$

where, Ω is a bounded convex polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$ with boundary $\partial\Omega$. Here, ν is the coefficient of kinematic viscosity and κ is the retardation time or the time of relaxation of deformations. In the context of viscoelastic

* Corresponding author.
E-mail addresses: ambit.pany@gmail.com (A.K. Pany), mymsciitm@gmail.com (S. Bajpai), akp@math.iitb.ac.in (A.K. Pani).

http://dx.doi.org/10.1016/j.cam.2016.01.037
0377-0427/

fluid, this model was first introduced by Pavlovskii [1], who called it as a model describing the motion of weakly concentrated water-polymer solution. It was called Kelvin–Voigt model by Oskolkov [2] and his collaborators. Subsequently, Cao et al. [3] proposed it as a smooth, inviscid regularization of the 2D and 3D-Navier–Stokes equations. For applications of such models in organic polymer and food industry, and in the mechanisms of diffuse axonal injury, etc., we refer to [4–6].

Earlier, based on the analysis of Ladyzhenskaya [7] in the context of Navier–Stokes equations, Oskolkov [8,9] have proved existence of a unique 'almost' classical solution in finite time interval for the problem (1.1)–(1.3). Subsequently, further investigations on solvability were continued by Oskolkov and his group members, see [10,11].

On numerical analysis of such problems, Oskolkov et al. [12] have discussed the convergence analysis of the spectral Galerkin approximation to (1.1)–(1.3) for all $t \geq 0$ assuming that the exact solution is asymptotically stable as $t \to \infty$. Subsequently, Pani et al. [13] have applied a variant of nonlinear semidiscrete spectral Galerkin method and optimal error estimates are proved. It is, further, shown that *a priori* error estimates are valid uniformly in time under uniqueness assumption. Recently, Bajpai et al. [14] have applied finite element Galerkin methods for the problem (1.1)–(1.3) with the forcing function $\mathbf{f} = 0$. They have proved *a priori* bounds for the exact solution in 3*D* and established exponential decay property. With an introduction of the Sobolev–Stokes projection, they have derived optimal error estimates, which again preserve the exponential decay property. In [15], completely discrete schemes which are based on both backward Euler and second order backward difference methods are analyzed and optimal error bounds which again preserve exponential decay property are established. For related articles in the context of Oldroyd viscoelastic model, we refer to [16–24].

In this paper, we, further, continue the investigation on finite element approximations to the problem (1.1)–(1.3) when the non-zero forcing function $\mathbf{f}$ belongs to $L^\infty(\mathbf{L}^2)$. This is crucial, particularly, in the study of the dynamical system (1.1)–(1.3), when the forcing function is assumed to be time independent. The major results obtained in this paper are summarized as follows:

(i) New regularity results for the solution of (1.1)–(1.3) even in 3*D*, which are valid uniformly in time are derived and as a consequence, existence of a global attractor is proved. It is further shown that these estimates hold uniformly in κ as $\kappa \mapsto 0$.
(ii) When $\mathbf{f}$ is independent of time, it is, further, established that the semi-discrete finite element method admits a discrete global attractor.
(iii) Based on the Sobolev–Stokes projection introduced earlier in [14], optimal error estimates for the semidiscrete Galerkin approximations to the velocity in $L^\infty(\mathbf{L}^2)$-norm as well as in $L^\infty(\mathbf{H}_0^1)$-norm and to the pressure in $L^\infty(L^2)$-norm are derived with error bounds depending on exponential in time.
(iv) Moreover, it is proved under uniqueness assumption that error estimates are valid uniformly in time.
(v) Under assumption $\kappa = O(h^{2\delta}),\ \delta > 0$ small, it is shown that the error analysis given in (iii)–(iv) yields quasi-optimal estimates.
(vi) Numerical experiments are conducted to confirm our theoretical findings. It is, further, established that the order of convergence does not deteriorate for small κ confirming results in (v).

Note that for (i), exponential weight functions in time are used which help us to derive regularity result for all $t > 0$. A special care is taken to show that these estimates are valid uniformly in κ as $\kappa \mapsto 0$. When $\mathbf{f}$ is independent of time, based on uniform estimates in time existence of a global attractor is shown for the semidiscrete scheme. For (iii), a use of Sobolev–Stokes projection as an intermediate projection helps us to retrieve optimal error estimates for the velocity vector in $L^\infty(\mathbf{L}^2)$-norm. When either $\mathbf{f} = 0$ or $\mathbf{f} = O(e^{-\alpha_0 t})$, we derive, as in [14], exponential decay property not only for the solution, but also for error estimates.

This paper is organized as follows. In Section 2, we discuss the weak formulation and state some basic assumptions. Section 3 is devoted to development of *a priori* bounds for the exact solutions. In Section 4, we describe the semidiscrete Galerkin approximations and derive *a priori* estimates with discrete global attractor for the semidiscrete solutions. In Section 5, we establish optimal error estimates for the velocity. Section 6 deals with the optimal error estimates for the pressure. In Section 7, results of numerical experiments, which confirm our theoretical estimates, are established.

2. Preliminaries and weak formulation

In this section, we define $\mathbb{R}^d,\ (d = 2, 3)$-valued function spaces using boldface letters as

$$\mathbf{H}_0^1 = (H_0^1(\Omega))^d, \qquad \mathbf{L}^2 = (L^2(\Omega))^d \quad \text{and} \quad \mathbf{H}^m = (H^m(\Omega))^d,$$

where $L^2(\Omega)$ is the space of square integrable functions defined in Ω with inner product $(\phi, \psi) = \int_\Omega \phi(x)\psi(x)\,dx$ and norm $\|\phi\| = \left(\int_\Omega |\phi(x)|^2\,dx\right)^{1/2}$. Further, $H^m(\Omega)$ denotes the standard Hilbert Sobolev space of order $m \in \mathbb{N}^+$ with norm $\|\phi\|_m = \left(\sum_{|\alpha|\le m}\int_\Omega |D^\alpha\phi|^2\,dx\right)^{1/2}$. Note that $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^{d}(\partial_j v_i, \partial_j v_i)\right)^{1/2} = \left(\sum_{i=1}^{d}(\nabla v_i, \nabla v_i)\right)^{1/2}.$$