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Positive semi-definiteness and sum-of-squares property of fourth order four dimensional Hankel tensors

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ABSTRACT

A symmetric positive semi-definite (PSD) tensor, which is not sum-of-squares (SOS), is called a PSD non-SOS (PNS) tensor. Is there a fourth order four dimensional PNS Hankel tensor? The answer for this question has both theoretical and practical significance. Under the assumptions that the generating vector $\mathbf{v}$ of a Hankel tensor $\mathcal{A}$ is symmetric and the fifth element v_4 of $\mathbf{v}$ is fixed at 1, we show that there are two surfaces M_0 and N_0 with the elements v_2, v_6, v_1, v_3, v_5 of $\mathbf{v}$ as variables, such that $M_0 \geq N_0$, $\mathcal{A}$ is SOS if and only if $v_0 \geq M_0$, and $\mathcal{A}$ is PSD if and only if $v_0 \geq N_0$, where v_0 is the first element of $\mathbf{v}$. If $M_0 = N_0$ for a point $P = (v_2, v_6, v_1, v_3, v_5)^\top$, there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors for such v_2, v_6, v_1, v_3, v_5. Then, we call such P a PNS-free point. We prove that a 45-degree planar closed convex cone, a segment, a ray and an additional point are PNS-free. Numerical tests check various grid points and report that they are all PNS-free.

1. Introduction

In 1888, young Hilbert [1] proved that for homogeneous polynomials, only in the following three cases, a positive semi-definite (PSD) form definitely is a sum-of-squares (SOS) polynomial: (1) $m = 2$; (2) $n = 2$; (3) $m = 4$ and $n = 3$, where m is the degree of the polynomial and n is the number of variables. Hilbert proved that in all the other possible combinations of n and even m, there are PSD non-SOS (PNS) homogeneous polynomials. The most well-known PNS homogeneous polynomial is the Motzkin function [2] with $m = 6$ and $n = 3$. Other examples of PNS homogeneous polynomials were found in [3–6].

A homogeneous polynomial is uniquely corresponding to a symmetric tensor [7]. For a symmetric tensor, m is its order and n is its dimension. One important class of symmetric tensors is the Hankel tensor. Hankel tensors have important applications in signal processing [8–10], automatic control [11], and geophysics [12,13]. For example, Papy et al. [14,15] proposed a novel Hankel tensor model to analyze time-domain signals in nuclear magnetic resonance spectroscopy, which is used for brain tumor detection [16]. A fast computational framework for products of a Hankel tensor and vectors is addressed in Ding et al. [17]. In geophysics, Trickett et al. [13] established a new multidimensional seismic trace interpolator by using Hankel tensors.

In mathematical science, Luque and Thibon [18] studied the Hankel hyperdeterminants. Xu [19] studied the spectra of Hankel tensors and gave some upper bounds and lower bounds for the smallest and the largest eigenvalues. In [20], two

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classes of PSD Hankel tensors were identified. They are even order strong Hankel tensors and even order complete Hankel tensors. It was proved in [21] that complete Hankel tensors are strong Hankel tensors, and even order strong Hankel tensors are SOS tensors. It was also shown there that there are SOS Hankel tensors and PSD Hankel tensors, which are not strong Hankel tensors. Thus, a question was raised in [21]: Are all PSD Hankel tensors SOS tensors [22,23]? If there are no PSD non-SOS Hankel tensors, the problem for determining a given even order Hankel tensor is PSD or not can be answered by solving a semi-definite linear programming problem [21,24,25].

We may call the problem raised by the above question as the Hilbert–Hankel problem. In a certain sense, it is the Hilbert problem with a Hankel constraint. According to Hilbert [1,6], one case with low values of m and n, in which there are PNS homogeneous polynomials, is that $m = 6$ and $n = 3$. In [26], the Hilbert–Hankel problem with order six and dimension three was studied. Four special cases were analyzed. Thousands of random examples were checked. No PNS Hankel tensors of order six and dimension three were found in [26]. Theoretically, it is still an open problem whether there are PNS Hankel tensors of order six and dimension three or not.

According to Hilbert [1,6], another case with low values of m and n, in which there are PNS homogeneous polynomials, is that $m = n = 4$. In this paper, we consider this special case in a Hankel context. Let $\mathbf{v} = (v_0, v_1, \ldots, v_{12})^\top \in \Re^{13}$. A fourth order four dimensional *Hankel tensor* $\mathcal{A} = (a_{i_1i_2i_3i_4})$ is defined by

$$a_{i_1i_2i_3i_4} = v_{i_1+i_2+i_3+i_4-4},$$

for $i_1, i_2, i_3, i_4 = 1, 2, 3, 4$. The corresponding vector $\mathbf{v}$ that defines the Hankel tensor $\mathcal{A}$ is called the *generating vector* of $\mathcal{A}$. For $\mathbf{x} = (x_1, x_2, x_3, x_4)^\top \in \Re^4$, a Hankel tensor $\mathcal{A}$ uniquely defines a Hankel polynomial

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes 4} = \sum_{i_1,i_2,i_3,i_4=1}^{4} a_{i_1i_2i_3i_4}x_{i_1}x_{i_2}x_{i_3}x_{i_4} = \sum_{i_1,i_2,i_3,i_4=1}^{4} v_{i_1+i_2+i_3+i_4-4}x_{i_1}x_{i_2}x_{i_3}x_{i_4}. \tag{1}$$

If $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Re^4$, the Hankel tensor $\mathcal{A}$ is called *positive semi-definite* (PSD). If $f(\mathbf{x})$ can be represented as a sum of squares of quadratic homogeneous polynomials, the Hankel tensor $\mathcal{A}$ is called *sum-of-squares* (SOS). Clearly, $\mathcal{A}$ is PSD if it is SOS.

In the next section, we present some necessary conditions for the positive semi-definiteness of fourth order four dimensional Hankel tensors.

We may see that the role of v_j is symmetric in $f(\mathbf{x})$. In Section 3, we assume that

$$v_j = v_{12-j} \tag{2}$$

for $j = 0, \ldots, 5$. Under this assumption, by the results of Section 2, if $\mathcal{A}$ is PSD, we have $v_0 = v_{12} \geq 0$ and $v_4 = v_8 \geq 0$. Moreover, if $v_4 = v_8 = 0$ and $\mathcal{A}$ is PSD, $\mathcal{A}$ is SOS. Thus, we may only consider the case that $v_4 = v_8 > 0$. Since $\mathcal{A}$ is PSD or SOS or PNS if and only if $\alpha\mathcal{A}$ is PSD or SOS or PNS respectively, where α is an arbitrary positive number, we may simply assume that

$$v_4 = v_8 = 1. \tag{3}$$

Next, we show that there is a function $\eta(v_5, v_6)$ such that $\eta(v_5, v_6) \leq 1$ if $\mathcal{A}$ is PSD. We propose that there are two functions $M_0(v_2, v_6, v_1, v_3, v_5) \geq N_0(v_2, v_6, v_1, v_3, v_5)$, defined for $\eta(v_5, v_6) < 1$, such that $\mathcal{A}$ is SOS if and only if $v_0 \geq M_0$, and $\mathcal{A}$ is PSD if and only if $v_0 \geq N_0$. If $M_0 = N_0$ for some v_2, v_6, v_1, v_3, v_5, then there are no fourth order four dimensional PNS Hankel tensors for such v_2, v_6, v_1, v_3, v_5 under the symmetric assumption (2). We call such a point $P = (v_2, v_6, v_1, v_3, v_5)^\top \in \Re^5$ a *PNS-free point* of fourth order four dimensional Hankel tensors, or simply a PNS-free point. We call the set of points in $\Re^5$, satisfying $\eta(v_5, v_6) < 1$, the *effective domain* of fourth order four dimensional Hankel tensors, or simply the effective domain, and denote it by S. We show that if all the points in S are PNS-free, then there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors.

In Section 4, we show that a point P in S is PNS-free if there is a value M, such that when $v_0 = M$, $f_0(\mathbf{x}) \equiv f(\mathbf{x})$ has an SOS decomposition, and $f_0(\bar{\mathbf{x}}) = 0$ for $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)^\top \in \Re^4$ with $\bar{x}_1^2 + \bar{x}_4^2 \neq 0$. We call such a value M, such an SOS decomposition of $f_0(\mathbf{x})$, and such a vector $\bar{\mathbf{x}}$ the *critical value*, the *critical SOS decomposition* and the *critical minimizer* of $\mathcal{A}$ at P, respectively. Then, we show that the segment $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$ is PNS-free. We conjecture that this segment is the minimizer set of both M_0 and N_0. Then, we show that the 45° planar closed convex cone $C = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, b, 0, 0, 0)^\top : a \geq b \geq 1\}$, the ray $R = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, 0, 0, 0, 0)^\top : a \leq 0\}$ and the point $A = (1, 0, 0, 0, 0)^\top$ are also PNS-free. We illustrate L, C, R and A in Fig. 1.

In Section 5, numerical tests check various grid points, and find that $M_0 = N_0$ there. Thus, they are also PNS-free. Therefore, numerical tests indicate that there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors.

Some final remarks are made in Section 6.