Journal of Computational and Applied Mathematics 301 (2016) 64–73

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Performance evaluation of output analysis methods in steady-state simulations

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ARTICLE INFO

*Article history:*
Received 23 June 2015
Received in revised form 5 December 2015

*Keywords:*
Steady-state simulation
Mean
Variance
Quantile
Output analysis methods

ABSTRACT

Output analysis methods of steady-state simulations have extensively been subject of study to evaluate the performance when estimating the mean. However, smaller efforts have been placed on performance evaluation of these methods to estimate variance and quantiles. In this paper, we empirically evaluate the performance of output analysis methods based on multiple replications and batches to estimate mean, variance and quantile with the same set of data. The evaluation of the performance of the methods is based on the empirical coverage of the true value using confidence intervals, the average bias, relative error and mean squared error. The methods are applied to estimate the average, variance and quantiles of waiting time in an $M/M/1$ queue. The results show that the methods based on non-overlapping batches perform consistently well in all the metrics. The performance of the other methods varies depending on the metric and the parameters of the simulation. In addition, we provide another example of a non-geometric ergodic Markov chain to show that asymptotically valid confidence intervals for quantiles can be obtained using batches and replications.



## 1. Introduction

Typically, researchers and analysts focus on the estimation of the mean in steady-state simulations. However, there are other performance measures that might be of interest, such as variance and quantiles, which are considered as measures of risk. Steady-state simulations are of particular interest for researchers due to the different approaches to analyze the output of this type of models and the variables that the analysts must decide on, such as number of replications/batches and run length. In addition, steady-state simulations are applied in a wide variety of queueing systems, such as communications (e.g., see [1]), production systems (e.g., see [2]), logistics (e.g., see [3]) and healthcare (e.g., see [4]).

In this paper we are interested in comparing the effectiveness of multiple replications with and without warm-up, non-overlapping batches and spaced-batches for estimating the mean, variance and quantile of a steady-state simulation. The results shown in this paper extend and correct the preliminary results published in [5,6]. In particular, we correct an error in the code that underestimated the coverage performance of the methods, we include the analysis of multiple replications with and without warm up period and we include the analysis of the mean squared error as a performance measure of the output analysis methods. In addition, we expand the analysis of quantile estimation in steady-state for a non-geometric ergodic Markov chain and empirically show that the validity of the batch-based method is valid under weaker conditions than geometric ergodicity.

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http://dx.doi.org/10.1016/j.cam.2016.01.039
0377-0427/

Multiple replications consist of running multiple independent runs of the same model [7]. Usually, multiple replications are accompanied with a warm-up period to reduce initialization bias [8]. Therefore, the data collected during the warm-up is discarded for analysis. However, warm up may not be needed if the initial conditions are well selected, as found in [9]. Given that the effect of the warm up depends on the replication length, we include in this paper the analysis of the performance using multiple replications with warm-up period (*MR*) and using multiple replications without warm-up ($MR_0$).

On the other hand, the method of non-overlapping batches (*B*) consists of running a single replication, which is divided in batches of equal size [7]. The batch size must be large enough to ensure almost independence of data. Large few batches are often preferred to small several batches because of correlation and normality requirements [10].

A different way to implement batch-based analysis is through spaced-batches (*SB*) which consist of having a space of observations between successive batches. The observations occurring in the spaces are not considered for the analysis. The purpose of *SB* is to reduce dependence amongst batches.

The effectiveness of multiple replications and batches has been widely discussed for estimating the mean in steady-state simulations. For example, batch means have been empirically shown to be superior to multiple replications in coverage and confidence interval's half-width [11]. The advantages of both methods have been combined in a methodology called replicated batch means designed to improve the coverage of (discrete state space) nearly decomposable Markov chains [12].

Estimation of variance has been discussed in methodologies for estimating a nonlinear function of a steady-state mean. These methodologies suggest using jackknife to reduce the bias and mean squared error of point estimators [13]. In addition, delta method has been used to show the asymptotic validity of confidence intervals based on the batch means method (see [13,14]).

Methods for steady-state quantile estimation have also been discussed in the literature. For example, a batch quantile methodology is discussed in [15–17]. A bias expansion for the jackknife, classical and batch means estimators for steady-state quantiles are provided in [18]. In addition, a simulation-based quantile estimator whose probability of not lying in a prespecified vicinity of the true quantile quickly converges to zero with the sample size is presented in [19]. Methods for estimating steady-state means have been modified to estimate steady-state quantiles, such as the sequential procedure proposed in [20].

Muñoz shows that the batch-based methods for quantile estimation are asymptotically valid under the assumption of geometric ergodicity of the underlying Markov chain (see [18]). In this paper, we successfully applied *MR* and *B* output analysis methods for estimating the quantiles of a Markov chain with non-geometric ergodicity. Our experiments suggest that these estimation methodologies may be asymptotically valid under weaker conditions.

## 2. Methodology

The simulation models used in our experiments were implemented in Excel using Visual Basic for Applications and a Mersenne twister pseudo-random number generator [21] coded in C++. Furthermore, the output analysis methods were implemented for allowing the estimation of all three performance measures using the same data of each run.

In order to make a fair comparison of the three output analysis methods under study, the experiments have the same number of observations. Thus, the *MR* (which includes warm-up), $MR_0$ (no warm-up), *SB* and *B* use the same simulation length for each scenario. Furthermore, the length of the warm-up period in *MR* is also the number of observations between consecutive batches in *SB*.

The formulations used for constructing the confidence intervals are asymptotically valid. These formulations are based on the following parameters:

$m$ = Total number of observations in the simulation experiment,
$n$ = Number of replications for *MR* and $MR_0$ or number of batches for *B* or *SB*,
$k$ = Number of observations per replication for $MR_0$ and *MR* (including warm-up), or number of observations per batch for *B* or *SB* (including space between batches), $m = kn$,
$d$ = Number of observations in the warm-up period for *MR* or number of observations in the space between batches for *SB*($d = 0$ for *B* and for $MR_0$),
$Y_{ij}$ = Value of the $j_{th}$ observation of replication $i$ for*MR* (batch $i$ for *B* or *SB*), $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k$.

The point estimators for the (steady-state) mean, variance and $\alpha$-quantile are defined by

$$\hat{\mu} = \frac{\sum_{i=1}^{n}\sum_{j=d+1}^{k} Y_{ij}}{n(m-d)}, \tag{1}$$

$$\hat{V} = \frac{1}{n(k-d)-1}\left[\sum_{i=1}^{n}\sum_{j=d+1}^{k} Y_{ij}^2 - n(k-d)\hat{\mu}^2\right], \tag{2}$$

$$\hat{q}_\alpha = X_{\lceil \alpha n(k-d)\rceil}, \tag{3}$$