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A semi-smooth Newton method for a special piecewise linear system with application to positively constrained convex quadratic programming

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ABSTRACT

In this paper a special piecewise linear system is studied. It is shown that, under a mild assumption, the semi-smooth Newton method applied to this system is well defined and the method generates a sequence that converges linearly to a solution. Besides, we also show that the generated sequence is bounded, for any starting point, and a formula for any accumulation point of this sequence is presented. As an application, we study the convex quadratic programming problem under positive constraints. The numerical results suggest that the semi-smooth Newton method achieves accurate solutions to large scale problems in few iterations.

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1. Introduction

In this paper we consider the following special piecewise linear system:

$$x^{+} + Tx = b, \tag{1}$$

where, denoting by $\mathbb{R}^{n\times n}$ the set of $n \times n$ matrices with real entries and $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$ the n-dimensional Euclidean space, the data consists of b a vector in $\mathbb{R}^n$, T a nonsingular matrix in $\mathbb{R}^{n\times n}$, the variable x is a vector in $\mathbb{R}^n$ and x^+ is the vector in $\mathbb{R}^n$ with ith component equal to $(x_i)^+ = \max\{x_i, 0\}$. In [1] was proposed a semi-smooth Newton's method for solving (1). Under suitable assumption was showed the finite convergence to a solution of (1). Some works dealing with (1) and its generalizations include [1–6]. It is worth mentioning that a similar equation has been studied in [7].

The purpose of the present paper is to discuss the semi-smooth Newton's method introduced in [1], to solve (1), under new assumptions. As an application, we use the obtained results to study the remarkable instance of (1),

$$[Q - I]\,x^{+} + x = -\tilde{b}, \tag{2}$$

where the data consists of Q a positive definite real matrix of size $n \times n$ and $\tilde{b} \in \mathbb{R}^n$. Moreover, we present some computational experiments designed to investigate its practical viability. It is worth pointing out that the semi-smooth Newton's

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method for solving (2) was studied in [8] and some computational tests were presented in [9]. The results obtained in this paper improve the ones of [8]. As we will show, the system (2) arises from the optimality condition of the convex quadratic programming problem under a positive constraint,

$$\begin{aligned} &\text{Minimize } \frac{1}{2}x^\top Qx + x^\top \tilde{b} + c \\ &\text{subject to } x \in \mathbb{R}^n_+, \end{aligned} \tag{3}$$

where c is a real number and $\mathbb{R}^n_+$ is the nonnegative orthant. Note that, without loss of generality, we can assume Q symmetric in (3) because the objective function of (3) is equal to $\frac{1}{2}x^\top \tilde{Q}x + x^\top \tilde{b} + c$, where $\tilde{Q} = \frac{1}{2}(Q + Q^T)$ is a symmetric matrix. Positively constrained convex quadratic programming is equivalent to the problem of projecting the point onto a simplicial cone. The interest in the subject of projection arises in several situations, having a wide range of applications in pure and applied mathematics such as Convex Analysis (see *e.g.*, [10]), Optimization (see *e.g.*, [11–14]), Numerical Linear Algebra (see *e.g.*, [15]), Statistics (see *e.g.*, [16–18]), Computer Graphics (see *e.g.*, [19]) and Ordered Vector Spaces (see *e.g.*, [20–24]). The projection onto a general simplicial cone is difficult and computationally expensive, this problem has been studied *e.g.*, in [25–29]. It is a special convex quadratic program and its KKT optimality conditions consist in a linear complementarity problem (LCP) associated with it, see *e.g.*, [30,28]. Therefore, the problem of projecting onto simplicial cones can be solved by active set methods [31–33,30] or any algorithms for solving LCPs, see *e.g.*, [31,30] and special methods based on its geometry, see *e.g.*, [28,30]. Other fashionable ways to solve this problem are based on the classical von Neumann algorithm (see *e.g.*, Dykstra algorithm [34,17,35]). Nevertheless, these methods are also quite expensive (see the numerical results in [36] and the remark preceding Section 6.3 in [37]).

Following the ideas of [7], we show that the approach using semi-smooth Newton's method, for solving (3), has potential advantages over existing methods. The main advantage appears to be the global, linear convergence and to achieve accurate solutions of large scale problems in few iterations. Our numerical results suggest, for a given class of problem, that the number of required iterations is almost unchanged. The numerical results also indicate a remarkable robustness with respect to the starting point.

The organization of the paper is as follows. In Section 1.1, some notations and preliminaries used in the paper are presented. In Section 2 we study the convergence properties of the semi-smooth Newton's method for solving (1). In Section 3 the results of Section 2 are applied to find a solution of (3). In Section 4 we present some computational tests. Some final remarks are made in Section 5.

1.1. Notations and preliminaries

In this subsection we present the notations and some auxiliary results used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the canonical inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. The ith component of a vector $x \in \mathbb{R}^n$ is denoted by x_i. We use the partial ordering for vectors, defined by $x \le y$ meaning $x_i \le y_i$, for all $i = 1, \ldots, n$. For $x \in \mathbb{R}^n$, $\operatorname{sgn}(x)$ will denote a vector with components equal to 1, 0 or -1 depending on whether the corresponding component of the vector x is positive, zero or negative. If $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then denote $a^+ := \max\{a, 0\}$, $a^- := \max\{-a, 0\}$ and x^+ and x^- the vectors with ith component equal to $(x_i)^+$ and $(x_i)^-$, respectively. From the definitions of x^+ and x^- we have $x = x^+ - x^-$, $\langle x^+, x^-\rangle = 0$ and $x^+, x^- \in \mathbb{R}^n_+$.

Lemma 1. *Let $x, y \in \mathbb{R}^n$. Then $\left\|y^+ - x^+ - \operatorname{diag}(\operatorname{sgn}(x^+))(y - x)\right\| \le \|y - x\|$.*

Proof. For each $i \in \{1, \ldots, n\}$, we have two possibilities:

(a) $x_i < 0$. In this case, $\operatorname{sgn}(x_i^+) = 0$. Thus, $|y_i^+ - x_i^+ - \operatorname{sgn}(x_i^+)(y_i - x_i)| = |y_i^+| \le |y_i - x_i|$.
(b) $x_i \ge 0$. In this case, $\operatorname{sgn}(x_i^+) = 1$. Hence, $|y_i^+ - x_i^+ - \operatorname{sgn}(x_i^+)(y_i - x_i)| = |y_i^+ - y_i| \le |y_i - x_i|$.

Combining (a) and (b) we have $(y_i^+ - x_i^+ - \operatorname{sgn}(x_i^+)(y_i - x_i))^2 \le (y_i - x_i)^2$, for all $i = 1, \ldots, n$, which implies the desired inequality. □

The matrix $\mathrm{I} \in \mathbb{R}^{n\times n}$ denotes the identity matrix. If $x \in \mathbb{R}^n$ then $\operatorname{diag}(x) \in \mathbb{R}^{n\times n}$ will denote a diagonal matrix with (i, i)-th entry equal to x_i, $i = 1, \ldots, n$. Denote $\|M\| := \max\{\|Mx\| : x \in \mathbb{R}^n,\ \|x\| = 1\}$ for any $M \in \mathbb{R}^{n\times n}$. The next useful result was proved in 2.1.1, page 32 of [38].

Lemma 2. *Let $E \in \mathbb{R}^{n\times n}$. If $\|E\| < 1$, then $E - \mathrm{I}$ is invertible and $\|(E - \mathrm{I})^{-1}\| \le 1/(1 - \|E\|)$.*

We end this section with the contraction mapping principle (see 8.2.2, page 153 of [38]).

Theorem 1 (*Contraction Mapping Principle*). *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that there exists $\lambda \in [0, 1)$ such that $\|\phi(y) - \phi(x)\| \le \lambda\|y - x\|$, for all $x, y \in \mathbb{R}^n$. Then there exists a unique $\bar{x} \in \mathbb{R}^n$ such that $\phi(\bar{x}) = \bar{x}$.*

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