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Numerical analysis for the wave equation with locally nonlinear distributed damping

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ABSTRACT

In this paper, we present spectral methods in order to solve wave equation subject to a locally distributed nonlinear damping. Thanks to the efficiency and the accuracy of spectral method, we can check that discrete energy decreases to zero as time goes to infinity, uniformly with respect to the mesh size when the damping is supported in a suitable subset of the domain of consideration. We prove the convergence of the full Fourier–Galerkin discretization. Thus, we apply our schemes to illustrate the uniform discrete energy decay rates of the solution for a wide range of damping functions.

1. Introduction

The aim of this paper is to study the efficiency and accuracy of spectral methods to solve wave equation subject to a locally distributed nonlinear damping. In particular, we want to check numerically the polynomial energy decay of the solution of the damped wave equation to zero as time goes to infinity when the damping is supported in a suitable subset of the domain of consideration.

When we use finite-difference space discretization, the corresponding semi-discrete energy do not decay exponentially and uniformly (with respect to the mesh size h) to zero as time goes to infinity (with the exception of damping coefficient uniformly positive on the full domain) as it is proved in [1,2].

Namely, consider the system corresponding to a linear damping term

$$\begin{cases} \alpha u_{tt} - \Delta u + a(x)u_t = 0, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \Gamma,\ t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is an open and bounded domain of $\mathbb{R}^n$ with $n \in \mathbb{N}$ and $a : \Omega \to \mathbb{R}$ is a bounded nonnegative function defined on ω a nonempty open subset of Ω, satisfying $a(x) > a_0 > 0$ on $\overline{\omega}$.

The coefficient α in front of the time derivative is introduced such that the equation is dimensionally consistent. Without loss of generality, we can set this coefficient $\alpha = 1$ (by changing time scale).

It is well known (see [3–5]) that the energy of the damped wave equation decreases as time increases. More precisely, the energy of solutions of (1.1) satisfies, for some $T_0 > 0$

$$E(t) \le E(0) \exp\left(1 - \frac{t}{t_0}\right), \quad t \ge T_0.$$

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Furthermore, it is known that if there is a positive constant C and time T such that

$$E(0) \leq C \left(\int_0^T \int_\Omega a(x)|u_t(t)|^2 dxdt, \right)$$

for every solution of (1.1)(cf. [6]), the exponential decay is obtained. This inequality, called observability inequality, is valid (for $n \geq 2$) if and only if the damping subset ω satisfies the geometric control condition of Bardos et al. (cf.[7]).

Stability for the wave equation

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad x \in \Omega, \ t > 0$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, has been studied for long time by many authors.

Concerning linear wave equations, Rauch and Taylor [8] are among the pioneers in investigating the long time behavior of weak solutions of the Cauchy problem for the wave equation on compact manifold without boundary, when $g(s) = s$, assuming that $a \in C^\infty$ is a bounded nonnegative function. We say that the Rauch–Taylor condition holds if there exists a time T_0 such that any geodesic (also called ray of the geometric optics) with length greater than T_0 meets the set where, roughly speaking, $a(x) > 0$. Later, Bardos et al. in [7] introduce generalized rays: if a ray arrives at the boundary transversally, its continuation is a reflected ray, and if a ray encounters the boundary at a not nondiffractive point, its continuation does not "feel" the boundary. Using a duality argument, the authors obtain exact controllability results for the whole H^s-scale of spaces of controls. They also prove the uniform stabilization by a dissipative boundary condition.

In this context, it is important to cite Komornik's book [9], who uses the strategy based on energy inequalities which leads to computable decay rates for dissipative systems, under some regularity assumptions imposed on the damping. The paper of Martinez [10] provides a general method, though they are not optimal for important cases.

In contrast with the majority of the papers written on the subject, the paper by Lasiecka and Tataru [11] makes no assumption on the damping at the origin, except for the continuity and monotonicity. This was the very first paper to establish optimal decay rates obeyed by the energy function, without any growth assumptions imposed at the dissipation at the origin. The main trust of that method is to reduce the problem of computations of decay rates for a PDE to solving an appropriate-explicitly given-ODE of monotone type. In Cavalcanti et al. [12] explicit decay rates are obtained. The theory presented allows to consider both superlinear and sublinear behaviors at the origin of the dissipation in the presence of unstructured sources. This is accomplished by following the method presented in [11].

Lasiecka and Toundykov, in their works [13,14], consider the wave equation with localized damping and source term. The dissipation acts on small subsets of the domain, near a portion of the boundary. They study the asymptotic behavior as t goes to infinity and related decay rates for the corresponding solutions. In [15], the author studies the rate of decay of solutions of the wave equation with localized nonlinear damping without any growth restriction and without any assumption on the dynamics. Providing regular initial data, the asymptotic decay rates are obtained by solving a nonlinear ODE. For all these reasons, we propose here a spectral method combined with an explicit in time scheme to solve the locally 2D damped wave equation. We emphasize that the damping term is not linear and that the damping coefficient can be null on a part (with non null measure) of the domain. In this work, we consider the damping function $g(x) = x^3$, but we emphasize that we can adapt the proof of the convergence result for any odd functions g satisfying the hypothesis (H.2) given below. We thus study the full discretization of this problem both in the space and time variables in the case where the domain Ω is a square. As usual, discretization in both space and time of evolution equations leads to unstable or conditionally stable schemes. By using energy method, we give sufficient conditions for stability and we prove the convergence of spectral approximate solutions toward those of the nonlinear damped wave equations as $h \to 0$. Of course the proof is valid in 1D and can be extended to the 3D cases but with more restrictive stability conditions (due to the Sobolev inclusion). Finally, we numerically illustrate the uniform (with respect to the mesh size) decay rates of the energy associated with the problem (2.1) for a wide range of damping functions and coefficients. Especially we first observe that the Chebyshev approximation is better suited than Fourier approximation (because of the treatment of the boundary conditions) and that the effect of damping is more effective if it is localized near the boundary of Ω rather that inside Ω.

Our paper is organized as follows. In Section 2, we briefly recall the main results about the wave equation with locally distributed nonlinear damping. In Section 3, we describe the numerical scheme and we remind the main results about Fourier approximation. Section 4 is devoted to the derivation of stability lemma and to the convergence result, we adapt to our case the proof of convergence theorem given in [16]. Section 5 contains few numerical experiments.

2. Wave equation with nonlinear damping

In this section, we are going to present some results concerning the existence and asymptotic behavior for the solutions of

$$\begin{cases} u_{tt} - \Delta u + a(x)g(u_t) = 0, & x \in \Omega, \ t > 0, \\ u = 0, & x \in \Gamma, \ t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is an open and bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}$, $a : \Omega \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions satisfying