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Stable Q_k-Q_{k-1} mixed finite elements with discontinuous pressure

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ABSTRACT

Mixed finite elements play a central role in many important CFD applications involving Stokes solvers and alike. A natural mixed finite element for the Stokes equations is the Q_k-Q_{k-1} element on rectangular grids, by which the velocity is approximated by continuous polynomials of separated degree k and the pressure is approximated by discontinuous polynomials of separated degree $k-1$. Such an element is, however, not stable. We propose in this paper three modified Q_k-Q_{k-1} elements with certain element-wise divergence-free property of velocity, where the pressure space is slightly restricted to subspaces in C^{-1}-Q_{k-1}, yet the optimal order of approximation is still retained. The stability and approximation analysis for the new elements are presented. Comprehensive numerical experiments are also conducted to confirm the theoretical analysis and to reveal the super-convergence for some of these new elements.

1. Introduction

Mixed finite elements play a central role in many important CFD applications involving Stokes solvers and alike, such as the Stokes equations, Navier–Stokes equations, Brinkman model, coupled Navier–Stokes/Darcy model, etc. The week solutions of the Stokes equations, the velocity and the pressure, belong to H^1 and L^2 spaces, respectively. A natural mixed finite element for the Stokes equations would thus be the pair of the continuous polynomials and the discontinuous polynomials of one degree less [1–9]. On rectangular grids, such an element, C^0–Q_k/C^{-1}–Q_{k-1}, is not stable no matter how large k is, as shown by Brezzi and Falk [10]. Stenberg and Suri showed in [11] the stability, but a sub-optimal order of approximation, of the Q_k–Q_{k-2} element for all $k \geq 2$ in R^2. Bernardi and Maday showed the stability and the optimal order of approximation for the Q_k–P_{k-1} element, cf. [12]. Ainsworth and Coggins proved the stability and the optimal order of approximation for the Taylor–Hood type of Q_k–Q_{k-1} element, where the pressure space is continuous too, cf. [13]. We study this classical, yet still interesting and important problem, the $Q_k - Q_{k-1}$ element, in this paper, focusing on discontinuous pressure elements.

For $k = 1$, it is known that the Q_1–Q_0 element is not stable because there is a global weakly-spurious mode in pressure in addition to the well-known checkerboard mode, cf. [14–17]. Many stabilized Q_1–Q_0 methods are proposed [15,18–21,17].

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For $k > 1$, as mentioned above, there are works on the Taylor–Hood type of Q_k–Q_{k-1} element and also works on the piecewise-constant enriched pressure Taylor–Hood element, [13,22–25]. However, for the most natural choice of discontinuous pressure elements, there is only one work, the so-called divergence-free finite elements, being studied in [26–28]. In such an element, the velocity is approximated by continuous $Q_{k,k-1} \times Q_{k-1,k}$ polynomials (degree k in x but degree $k-1$ in y for the first component of velocity) while the pressure is approximated by discontinuous polynomials $Q_{k-1}^- = \operatorname{div}(Q_{k,k-1} \times Q_{k-1,k})$, i.e., discontinuous Q_{k-1} polynomials with one constraint at each vertex, see (2.3) in Section 2 for the precise definition where $Q_{k-1}^- = P_h'$. We note that, by choosing the divergence space, $\operatorname{div}(Q_{k,k-1} \times Q_{k-1,k})$, as the discrete finite element space for the pressure, the spurious modes in discontinuous Q_{k-1} space are filtered out automatically. On the other side, the discrete velocity solution is divergence-free if and only if the discrete pressure space is the divergence of the discrete velocity space. Because the divergence of a $Q_{k,k-1} \times Q_{k-1,k}$ vector is a Q_{k-1} polynomial, this element makes a perfect match for the two discrete spaces and produces divergence-free solutions for the velocity.

Along this line, we study in this paper the stable and discontinuous pressure Q_k–Q_{k-1} elements. For the discrete velocity space, however, instead of using the subspace $Q_{k,k-1} \times Q_{k-1,k}$ as in [26–28], we insist using the whole Q_k space. As the element of full Q_k–Q_{k-1} space is not stable [10], we stabilize it by reducing the pressure space C^{-1}–Q_{k-1}, i.e., choose a subspace of C^{-1}–Q_{k-1} for the pressure discretization, while still trying to retain certain element-wise divergence-free property of velocity. We will consider three such subspaces and prove the stability and the optimal order of approximation for the three new elements. Comprehensive numerical experiments on these elements will also be presented to confirm the theoretical analysis as well as reveal the super-convergence for some of these new elements.

2. The mixed elements

We consider a model stationary Stokes problem: Find the velocity $\mathbf{u}$ and the pressure p on a two-dimensional polygonal domain Ω which can be subdivided into rectangles, such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The standard variational problem for (2.1) reads: Find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega) := L^2(\Omega)/C = \{p \in L^2 | \int_\Omega p = 0\}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in L_0^2(\Omega). \end{aligned} \tag{2.2}$$

Here $H_0^1(\Omega)^2$ is the subspace of the Sobolev space $H^1(\Omega)^2$ (cf. [1]) with zero boundary trace, and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx, \\ b(\mathbf{v}, p) &= -\int_\Omega \operatorname{div} \mathbf{v}\, p\, dx, \\ (\mathbf{f}, \mathbf{v}) &= \int_\Omega \mathbf{f}\, \mathbf{v}\, dx. \end{aligned}$$

Let $\mathcal{T}_h$ be a rectangular grid on Ω:

$$\mathcal{T}_h = \left\{ K = [x_a, x_b] \times [y_c, y_d] \mid \cup K = \overline{\Omega},\ h_K = \max\{x_b - x_a, y_d - y_c\} \le h \right\}.$$

Let $Q_{k,l}$ be the space of polynomials of degree k in the first variable and of degree l in the second variable: $\sum_{i \le k, j \le l} c_{ij} x^k y^l$, and denote $Q_k = Q_{k,k}$.

Recall that for the divergence-free elements in [26–28], the velocity space is defined as $\mathbf{V}_h' = C^0$–$Q_{k,k-1} \times Q_{k-1,k}$ and the pressure space is defined as

$$P_h' = \left\{ p_h \in L_0^2(\Omega) \mid p_h|_K \in Q_{k-1},\ T_{\mathbf{x}_j} v_h = 0\ \mathbf{x}_j \text{ is a vertex of } \mathcal{T}_h \right\}, \tag{2.3}$$

where operator $T_{\mathbf{x}_j}$ is defined for each vertex $\mathbf{x}_j$ of $\mathcal{T}_h$, see Fig. 2.1,

$$T_{\mathbf{x}_j} p_h = \begin{cases} p_h|_{K_1}(\mathbf{x}_j) & \mathbf{x}_j \text{ in Fig. 2.1(a)}, \\ p_h|_{K_1}(\mathbf{x}_j) - p_h|_{K_2}(\mathbf{x}_j) & \mathbf{x}_j \text{ in Fig. 2.1(b)}, \\ \displaystyle\sum_{i=1}^{4} (-1)^{i-1} p_h|_{K_i}(\mathbf{x}_j) & \mathbf{x}_j \text{ in Fig. 2.1(c)}. \end{cases} \tag{2.4}$$