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Symbolic computation of Drazin inverses by specializations

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ABSTRACT

In this paper, we show how to reduce the computation of Drazin inverses over certain computable fields to the computation of Drazin inverses of matrices with rational functions as entries. As a consequence we derive a symbolic algorithm to compute the Drazin inverse of matrices whose entries are elements of a finite transcendental field extension of a computable field. The algorithm is applied to matrices over the field of meromorphic functions, in several complex variables, on a connected domain and to matrices over the field of Laurent formal power series.

1. Introduction

This paper shows how symbolic computation can be applied to compute Drazin inverses of matrices whose coefficients are rational functions of finitely many transcendental elements over the field $\mathbb{C}$ of the complex numbers; we recall that an element a is transcendental over a field $\mathbb{K}$ if a is the root of none (non-zero) univariate polynomial, with coefficients in $\mathbb{K}$ (see e.g. pg. 114 in [1]).

1.1. The interest of Drazin inverses

Motivated on the notion of generalized inverse introduced by Moore and Penrose, Drazin, in [2], introduced the notion of Drazin inverse in the more general context of rings and semigroups. For matrices, Drazin inverse is defined as follows. Let A be a square matrix over a field, then the Drazin index of A is the smallest non-negative integer k such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$; let us denote it by $\mathrm{index}(A)$. In this situation, the Drazin inverse of A is the unique matrix satisfying the following matrix equations:

$$\begin{cases} A^{\mathrm{index}(A)+1} \cdot X = A^{\mathrm{index}(A)}, \\ A \cdot X = X \cdot A, \\ X \cdot A \cdot X = X. \end{cases} \tag{1}$$

Many authors have analyzed the properties of Drazin inverses (see e.g. Chapter 4 in [3], Chapters 7 in [4–6]) as well as their applications (see e.g. Chapters 8 and 9 in [7,4,8,9]); particularly interesting are the application to singular differential equations and difference equations, and to finite Markov chains.

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1.2. Computing Drazin inverses: the state of the art

An important alternative issue in the topic is the computation of the Drazin inverse (see e.g. [3,4,7,6,10–15]). The problem has been approached mainly for matrices with complex numbers. Nevertheless, in a second stage, different authors have addressed the problem of computing Drazin inverses of matrices whose entries belong to other fields as, for instance, rational function fields (see [16–19]). Furthermore, symbolic techniques have proven to be a suitable tools for this goal.

The next challenging step is the computation of the Darzin inverses over more general domains, as for instance the field of formal Laurent series (see connection to the study of analytically perturbed matrices in [7]) or the field of meromorphic functions; in [20] one can read about the difficulties of reasoning with transcendental functions. In this paper, we show how to reduce the symbolic computation of the Drazin inverses over such domains to the computation over the field of rational functions, and hence the already available algorithms, in particular the one based on Gröbner bases developed in [19], can be applied.

1.3. Main contributions of the paper

In general terms, the main contribution of the paper is the establishment of an algorithmic criterium to symbolically reduce the computation of Darzin inverses of matrices over a field $\mathbb{C}(t_1, \ldots, t_r)$, where $t_1, \ldots, t_r$ are transcendental elements over $\mathbb{C}$, to the computation of Darzin inverses of matrices over the field of rational functions $\mathbb{C}(w_1, \ldots, w_r)$, where $w_1, \ldots, w_r$ are independent complex variables. More precisely, the main contributions of the paper can be summarized as follows:

- we prove the existence, and actual computation, of a multivariate polynomial in $\mathbb{C}[w_1, \ldots, w_r]$ (that we call the evaluation polynomial, see Definition 7) such that if it does not vanish at $(t_1, \ldots, t_r)$ the computation of the Drazin inverse over $\mathbb{C}(t_1, \ldots, t_r)$ is reduced to the computation of the Drazin inverse over $\mathbb{C}(w_1, \ldots, w_r)$.
- As an application, we show how to compute the Drazin inverse of matrices whose entries are meromorphic functions or Laurent formal power series.
- Furthermore, we show how to relate the specialization of the Drazin inverse of a matrix, with meromorphic function entries, and the Drazin inverse of the specialization.
- Also, as a consequence of these ideas, one gets a method to compute the Laurent expansion of the Drazin inverses of analytically perturbed matrices.

1.4. Intuitive idea of the novel proposed solution method

Given a matrix A, the idea consists in the following three steps:

1. [Specialization step] we associate to the matrix A a new matrix A^*, whose entries are rational functions in several variables;
2. [Inverse computation step] secondly we compute the Drazin inverse of A^*, for instance using the algorithm in [19];
3. [Evaluation step] finally, from the Drazin inverse of A^* we get the Drazin inverse of A.

More precisely, we consider a computable field $\mathbb{K}$, that is a field for which all operations can be executed by means of an algorithm (see pg. 16 in [21]). In addition we consider the chain of fields

$$\mathbb{K} \subset \mathbb{K}(t_1, \ldots, t_r) \subset \mathbb{F}$$

where $\mathbb{K}(t_1, \ldots, t_r)$ is the field extension of $\mathbb{K}$ via the adjunction of finitely many elements $t_1, \ldots, t_r \in \mathbb{F} \setminus \mathbb{K}$; we recall that the extension of $\mathbb{K}$ via the adjunction of $\{t_1, \ldots, t_r\}$ is the intersection of all fields containing $\mathbb{K}$ and $\{t_1, \ldots, t_r\}$ or equivalently the field of all rational combinations of the elements in $\{t_1, \ldots, t_r\}$ with the elements in $\mathbb{K}$ (see e.g. Section 6.2. in [1]). For instance, we may take

$$\mathbb{C} \subset \mathbb{C}(\sin(z), \cosh(z)) \subset \mathrm{Mer}(\mathbf{z}, \Omega)$$

where $\mathrm{Mer}(\mathbf{z}, \Omega)$ is the field of meromorphic functions on a connected domain Ω of $\mathbb{C}$, or we may take

$$\mathbb{C} \subset \mathbb{C}\left(\sum_{n=-2}^{\infty} \frac{1}{n^4+1} z^n, \sum_{n=0}^{\infty} \frac{n}{n+1} z^n\right) \subset \mathbb{C}((z))$$

where $\mathbb{C}((z))$ is the field of Laurent formal power series. In this situation, a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{K}(t_1, \ldots, t_r))$, with entries in the intermediate field $\mathbb{K}(t_1, \ldots, t_r)$, is given, and we want to compute the Drazin inverse of A. For this purpose, we replace A by a matrix $A^* \in \mathcal{M}_{n\times n}(\mathbb{K}(w_1, \ldots, w_r))$ with entries in the field of rational functions $\mathbb{K}(w_1, \ldots, w_r)$. Then, the Drazin inverse of A is achieved from the Drazin inverse of A^* by specializing $(w_1, \ldots, w_r)$ at $(t_1, \ldots, t_r)$. We prove the existence, and we actually show how to compute, of a multivariate polynomial in $\mathbb{K}[w_1, \ldots, w_r]$ such that if it does not vanish at $(t_1, \ldots, t_r)$ the correctness of the method is guaranteed.