Journal of Computational and Applied Mathematics 301 (2016) 259–270

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A regularized sampling algorithm for reconstructing non-bandlimited signals

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ARTICLE INFO

Article history:
Received 27 November 2014
Received in revised form 11 September 2015

Keywords:
Non-bandlimited sampling
Ill-posedness
Regularization

ABSTRACT

In this paper the reconstruction of non-bandlimited sampling is discussed and a regularized sampling algorithm for non-bandlimited signals is presented. The error of the regularized sampling algorithm is presented and compared with the previous algorithm based on Shannon's sampling theorem and compared with the Tikhonov regularization method in the noisy case.

1. Introduction

In this section, we describe the bandlimited signals and the sampling theorem.

Definition. A function $f \in L^2(\mathbf{R})$ is said to be Ω-*bandlimited* if

$$\hat{f}(\omega) = 0, \quad \forall \omega \notin [-\Omega, \Omega].$$

Here $\hat{f}$ is the *Fourier transform* of f:

$$\mathbf{F}(f)(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{i\omega t} dt, \quad \omega \in \mathbf{R}. \tag{1}$$

We then have the inversion formula:

$$\mathbf{F}^{-1}(\hat{f})(t) = f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{-i\omega t} d\omega, \quad \text{a.e. } t \in \mathbf{R}. \tag{2}$$

For bandlimited signals, we have the following sampling theorem [1].

Shannon Sampling Theorem. The Ω-bandlimited signal $f(t)$ can be exactly reconstructed from its samples $f(nh)$, and

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\sin \Omega(t - nh)}{\Omega(t - nh)} f(nh)$$

where $h := \pi/\Omega$.

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http://dx.doi.org/10.1016/j.cam.2015.11.032
0377-0427/

In the case the signal is not bandlimited, the Shannon Sampling Theorem is not satisfied. In [2], we have an error estimation if we use the sampling formula in the Shannon Sampling Theorem for non-bandlimited signals:

$$\left|\sum_{n=-\infty}^{\infty}\frac{\sin\Omega(t-nh)}{\Omega(t-nh)}f(nh)-f(t)\right|\leq\frac{1}{\pi}\int_{|\omega|>\Omega}|\hat{f}(\omega)|d\omega.$$

However, this error estimation is not effective in practice due to the noise, since this is a highly ill-posed problem [3].

The sampling theory for non-bandlimited signals is also studied in [4–6]. However the ill-posedness and noise are not considered. So the methods in those papers are not reliable for a practical purpose. In this paper the ill-posedness is considered and the noise is in control by regularization. So the algorithm is more effective in practical applications.

In many practical problems, the samples $\{f(nh)\}$ are noisy:

$$f(nh)=f_T(nh)+\eta(nh),$$

where $\{\eta(nh)\}$ is the noise and $f_T\in L^2$ is the exact non-bandlimited signal. The ill-posedness of the Generalized Sampling Theorem [7] is discussed in [8,9]. The ill-posedness of the formula in the Shannon's Sampling Theorem can be seen in Section 4 in this paper.

Based on the regularized Fourier transform [10]

$$\hat{f}_\alpha(\omega)=\int_{-\infty}^{\infty}\frac{f(t)e^{i\omega t}dt}{1+2\pi\alpha+2\pi\alpha t^2},$$

(where $\alpha>0$ is the regularization parameter) and the Sampling Theorem, we construct the regularized sampling formula:

$$f_\alpha(t)=\sum_{n=-\infty}^{\infty}\frac{\sin\Omega(t-nh)}{\Omega(t-nh)}\frac{f(nh)}{1+2\pi\alpha+2\pi\alpha(nh)^2}.$$

In [11], an error estimation of the regularized sampling algorithm is given for bandlimited signals. In this paper, we will take the ill-posedness into account by the regularized sampling formula and give an error estimation of the regularized sampling algorithm for non-bandlimited signals.

Since the exact signal f_T is in L^2, we can assume the signal is ϵ-concentrated:

$$\int_{|t|\geq T}|f_T(t)|^2dt\leq\epsilon.$$

We also assume

$$\int_{|\omega|>\Omega}|\hat{f}_T(\omega)|d\omega\leq\epsilon$$

and

$$|\hat{f}_T(\omega)|\leq M=const.>0.$$

In Section 2, we give an error estimation of the regularized sampling formula for non-bandlimited signals. In Section 3, a more precise error estimation is given. In Section 4, we give numerical results of the Shannon's sampling formula and regularized sampling algorithm. Finally in Section 5, the regularized sampling algorithm is compared with the Tikhonov regularization method.

2. The error estimation of the regularized sampling algorithm

In order to give the error estimation of this regularized sampling formula, we suppose $\|\eta\|_\infty\leq\delta$ and we need the following lemmas.

Lemma 1.

$$\hat{K}(\omega):=\mathbf{F}\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2}\right]=\frac{1}{2a\alpha}e^{-a|\omega|}$$

where $a=(\frac{1+2\pi\alpha}{2\pi\alpha})^{\frac{1}{2}}$.

The proof is in the Appendix.

Lemma 2.

$$\hat{f}_{T\alpha}(\omega):=\mathbf{F}\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2}f_T(t)\right]=\frac{1}{4\pi a\alpha}\int_{-\infty}^{\infty}\hat{f}_T(u)e^{-a|u-\omega|}du.$$

The proof is in the Appendix.