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Efficient method for the computation of oscillatory Bessel transform and Bessel Hilbert transform[☆]

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ABSTRACT

In this paper, we study the numerical methods for the evaluation of two kinds of highly oscillatory Bessel transforms. Firstly, we rewrite both integrals as the sum of two integrals. By rewriting the Bessel function as a linear combination of Whittaker W function, we then transform one of integrals into the Fourier type, which can be transformed into the integrals on $[0, +\infty)$, and can be computed by some proper Gaussian quadrature, which take into account the asymptotic property of Whittaker W function as $x \to 0$. The other can be efficiently computed based on the evaluation of special functions. In addition, error analysis for the presented methods is given. Moreover, we also give an explicit formula for the integral $\int_0^{+\infty} \frac{J_\nu(\omega x)}{x-\tau}\,dx$ in terms of Meijer G-function, and then apply the method for the oscillatory Bessel transforms to the computation of highly oscillatory Bessel Hilbert transforms. Theoretical results and numerical examples demonstrate the efficiency and accuracy of the proposed methods.

1. Introduction

In this work, we are concerned with the problem of evaluating the highly oscillatory Bessel transforms of the form

$$I_1[f] = \int_0^a f(x)J_\nu(\omega x)\,dx \quad \text{and} \quad I_2[f] = \int_0^{+\infty} f(x)J_\nu(\omega x)\,dx, \tag{1.1}$$

where $a > 0$, ν is an arbitrary nonnegative real number, and $J_\nu(\omega x)$ is the Bessel function of the first kind of order ν. It is well known that the integrals (1.1) play an important role in many areas of science and engineering, for example, in astronomy, optics, quantum mechanics, seismology image processing, electromagnetic scattering [1–4]. Both integrals share the property that the larger the ω, the more oscillatory the integrands. Due to this property, when $\omega \gg 1$, a prohibitively large number of quadrature points are needed for the numerical computation of both integrals if one use classical numerical methods, such as, Simpson rule, Gaussian quadrature, etc.

In the last decades, there has been tremendous interest in developing numerical methods for the integral $\int_a^b f(x)J_\nu(\omega x)\,dx$ with $a \geq 0$. For the case $a > 0$, lots of methods have been devised, such as Levin method [5,6], Levin-type method [7], generalized quadrature rule [8,9], Filon-type method [10], Gauss–Laguerre quadrature [11–13]. For the case $a = 0$, a

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modified Clenshaw–Curtis method was presented by truncating $f(x)$ by its Chebyshev series approximation in [14]. Based on a special Hermite interpolant of $f(x)$ at the Clenshaw–Curtis points and the fast computation of modified moments, a Clenshaw–Curtis–Filon-type method was introduced in [15]. Recently, Chen [16] rewrote the integral in Fourier type, and then transformed the integral into the forms on $[0, +\infty)$ so that the integrand does not oscillate and decays exponentially fast. Consequently, the integral can be efficiently computed by using the Gauss–Laguerre quadrature. However, this method cannot be applied to the case that ν is not a nonnegative integer and the transformed Fourier type integrals are complex.

For the integral $\int_0^{+\infty} f(x)J_\nu(\omega x)\,dx$, asymptotic expansions have been given recently by several authors, one can refer to [17–19] to obtain more details. Also, Wong [20] given an explicit expression for the error term associated the expansion of the integral, from which an error bound can be obtained readily. For the numerical evaluation of this integral, as early as in 1982, Wong [21] constructed a Gaussian quadrature by rewriting it as a linear combination of the integrals of Hankel functions, which yields two line integrals with a positive weight function related to the modified Bessel function of the second kind of order ν. Recently, Asheim and Huybrechs [22] also constructed a Gaussian rule with Bessel function as its weight based on Gram–Schmidt orthogonalization. Unfortunately, the use of moments is numerically problematic in as much as they give rise to severe ill-conditioning [23]. For other methods, we refer the readers to [24] for a more general review.

In the present work, based on the following important identity [25]

$$J_\nu(z) = \frac{1}{(2\pi z)^{1/2}}\left\{ e^{\frac{1}{2}\left(\nu+\frac{1}{2}\right)\pi i} W_{0,\nu}(2iz) + e^{-\frac{1}{2}\left(\nu+\frac{1}{2}\right)\pi i} W_{0,\nu}(-2iz)\right\}, \tag{1.2}$$

where $W_{\kappa,\mu}(x)$ denotes the Whittaker W function [26], and the similar idea to [16,27], we present the new methods for both integrals in (1.1) by using generalized Gauss–Laguerre quadrature and logarithmic Gauss–Laguerre quadrature [28], which take into account of the asymptotic property of Whittaker W function as $x \to 0$.

In three dimensions water-wave radiation problem [29], one often come across the numerical computation of oscillatory Bessel Hilbert transform of the following form:

$$\Phi_n^m(\rho, y) = ⨍_0^{+\infty} k^n e^{-ky} J_m(k\rho)\frac{dk}{k-K}, \tag{1.3}$$

where $\rho > 0$. For more details, one can refer to [29].

In this paper, we are also concerned with the computation of oscillatory Bessel Hilbert transform

$$H^+(f(x)J_\nu(\omega x))(\tau) = ⨍_0^{+\infty} \frac{f(x)}{x-\tau} J_\nu(\omega x)\,dx, \tag{1.4}$$

where $0 < \tau < +\infty$. The asymptotics and numerical methods of this integral have been investigated in [30]. For the computation of the integral, the authors make the following decomposition

$$\begin{aligned} H^+(f(x)J_\nu(\omega x))(\tau) = {} & \int_0^a \frac{f(x)-f(\tau)}{x-\tau} J_\nu(\omega x)\,dx + \int_a^{+\infty} \frac{f(x)-f(\tau)}{x-\tau} J_\nu(\omega x)\,dx \\ & + f(\tau)\left\{ \int_0^a \frac{J_\nu(\omega x)}{x-\tau}\,dx + ⨍_a^{+\infty} \frac{J_\nu(\omega x)}{x-\tau}\,dx \right\}, \end{aligned} \tag{1.5}$$

with $0 < a < \tau$. However, the choice of a is depended on τ, and when τ is close to the origin, the methods are infeasible.

In this work, we rewrite the integral (1.4) as

$$H^+(f(x)J_\nu(\omega x))(\tau) = \int_0^{+\infty} \frac{f(x)-f(\tau)}{x-\tau} J_\nu(\omega x)\,dx + f(\tau) ⨍_0^{+\infty} \frac{J_\nu(\omega x)}{x-\tau}\,dx, \tag{1.6}$$

where, the first integral can be evaluated by the method for the integral $I_2[f]$, the second integral can be computed by explicit expression which will be given in Section 4.

This paper is organized as follows. In Section 2, we transform both integrals into the Fourier type, and construct some proper Gaussian quadrature rules to evaluate these Fourier type integrals. In Section 3, we also construct an alternative quadrature formula for the integrals with a logarithmic singularity. Error analysis is presented in Section 4, and numerical examples are also given to show the accuracy and efficiency of the presented methods. In Section 5, we apply methods presented in Section 4 to the computation of highly oscillatory Bessel Hilbert transforms, based on the fast computation of the integral

$$⨍_0^{+\infty} \frac{J_\nu(\omega x)}{x-\tau}\,dx.$$

2. Numerical methods for the integrals in (1.1)

In this section, we study the numerical methods for the integrals in (1.1). Throughout the paper, we do not distinguish the different constants C and R, and let $\log(z) = \log(|z|) + i\arg(z)$ denote the principal value of the logarithm.