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A high-order discontinuous Galerkin method for Itô stochastic ordinary differential equations

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The first author would like to dedicate this paper to his Father, Ahmed Baccouch, who unfortunately passed away during the completion of this work. Without his father's support and encouragement, he definitely would not become a professor of mathematics

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ABSTRACT

In this paper, we develop a high-order discontinuous Galerkin (DG) method for strong solution of Itô stochastic ordinary differential equations (SDEs) driven by one-dimensional Wiener processes. Motivated by the DG method for deterministic ordinary differential equations (ODEs), we first construct an approximate deterministic ODE with a random coefficient on each element using the well-known Wong–Zakai approximation theorem. Since the resulting ODE converges to the solution of the corresponding Stratonovich SDE, we apply a transformation to the drift term to obtain a deterministic ODE which converges to the solution of the original SDE. The corrected equation is then discretized using the standard DG method for deterministic ODEs. We prove that the proposed stochastic DG (SDG) method is equivalent to an implicit stochastic Runge–Kutta method. Then, we study the numerical stability of the SDG scheme applied to linear SDEs with an additive noise term. The method is shown to be numerically stable in the mean sense and also A-stable. As a result, it is suitable for solving stiff SDEs. Moreover, the method is proved to be convergent in the mean-square sense. Numerical evidence demonstrates that our proposed DG scheme for SDEs with additive noise has a strong convergence order of $2p + 1$, when p-degree piecewise polynomials are used. When applied to SDEs with multiplicative noise, the SDG method is strongly convergent with order p. Several linear and nonlinear test problems are presented to show the accuracy and effectiveness of the proposed method. In particular, we demonstrate that our proposed scheme is suitable for stiff stochastic differential systems.

1. Introduction and problem statement

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process. SDEs are powerful tools to model real problems with uncertainty. They have many applications in various branches of applied sciences, including economics, finance, insurance, biology, population dynamics, chemistry, epidemiology, physics, engineering, and aerospace. For instance, SDEs are used to model diverse phenomena such as fluctuating stock prices or physical system subject to thermal fluctuations. Typically, SDEs incorporate white noise which can be thought of as the derivative of Brownian motion (or Wiener process). We refer

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the reader to [1–3] and the references therein for further detailed discussion about properties of SDEs, their applications, and for a list of important citations on theoretical results for SDEs, including stability analysis.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space which consists of an arbitrary set Ω called the sample space, a σ-algebra $\mathcal{F}$ of subsets of Ω called events, and a probability measure P called its probability. We assume that $(\Omega, \mathcal{F}, P)$ is equipped with a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ satisfying the usual conditions (that is, it is right continuous and increasing family of sub-σ-algebras of $\mathcal{F}$ while $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$). Let $W = \{W(t),\ 0 \le t \le T\}$ be a given Brownian motion on this probability space.

In this paper, we propose a stochastic discontinuous Galerkin (SDG) method for strong solution of the following one-dimensional initial-value problem (IVP) for the SDE of Itô type driven by a Brownian motion:

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \quad t \in [0, T], \qquad X(0) = X_0, \tag{1.1}$$

where the initial condition X_0 is a real random variable, $X = \{X(t) : t \in [0, T]\}$ is a stochastic process (Itô process), $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a drift coefficient, $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a diffusion coefficient, and $W(t)$ represents the one-dimensional real-valued standard Wiener process (or Brownian motion), whose increment $\Delta W = W(t + h) - W(t)$ is a Gaussian random variable $N(0, h)$, where h is a constant step size. $W(t)$ can be interpreted in such a way that the derivative of W is the Gaussian white noise process (so that $W(t)$ is not differentiable). The noise is called additive if $b(t, x)$ does not depend on x; otherwise it is called multiplicative.

We will assume that the initial value X_0 is $\mathcal{F}_0$-measurable random variable in $\mathbb{R}$, independent of the Wiener process $W(t),\ t \in [0, T]$, and has a finite second moment *i.e.*, $\mathrm{E}\left[|X_0|^2\right] < \infty$, where $\mathrm{E}[\cdot]$ denotes the expected value. By a solution to the SDE (1.1) we mean a continuous $\mathcal{F}_t$-adapted process $X(t),\ t \in [0, T]$ satisfying (1.1) with probability 1. In this paper, we assume that the drift and diffusion functions satisfy the regularity properties that guarantees the existence and uniqueness of a global strong solution for (1.1). In particular, it suffices that the continuous functions a and b satisfy the global Lipschitz condition and the linear growth condition, that is, (i) there exists a positive constant L such that

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \le L|x - y|, \quad \forall\, t \in [0, T],\ \forall\, x, y \in \mathbb{R}, \tag{1.2}$$

and (ii) for all $t \ge 0$ and $x \in \mathbb{R}$, there is a constant $K > 0$ such that the following linear growth bound holds:

$$|a(t, x)| + |b(t, x)| \le K(1 + |x|). \tag{1.3}$$

It is well-known (see, *e.g.* Mao [4]) that under these hypothesis, (1.1) has a unique strong solution. Moreover, $\mathrm{E}\left[\sup_{0\le t\le T} |X(t)|^2\right] < \infty$. We note that if $a(t, 0)$ and $b(t, 0)$ are bounded on $[0, T]$ then (1.3) follows from (1.2) since $|a(t, x)| + |b(t, x)| \le |a(t, 0)| + |b(t, 0)| + |a(t, x) - a(t, 0)| + |b(t, x) - b(t, 0)| \le \max_{t\in[0,T]} |a(t, 0)| + \max_{t\in[0,T]} |b(t, 0)| + L|x - 0| \le K(1 + |x|)$.

The SDE (1.1) is, in fact, only a symbolic representation for the Itô stochastic integral equation

$$X(t) = X_0 + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s), \quad t \in [0, T], \tag{1.4}$$

where the first integral is pathwise deterministic Riemann integral and the second is an Itô stochastic integral with respect to the Wiener process $W(t)$. The latter integral cannot be defined pathwise as a deterministic Riemann–Stieltjes integral because the sample paths of the Wiener process, though continuous, are not differentiable or even of bounded variation on any finite interval.

Unlike deterministic ODEs, SDEs are complex, mostly because the white noise is almost everywhere discontinuous and has infinite variation [1]. Analytic solutions of SDEs are more difficult to obtain than in the deterministic case. Unfortunately, in many cases, analytic solutions of SDEs of the form (1.1) are not available and we are forced to use numerical methods to approximate them. Designing high-order numerical methods for SDEs is a recent area of research in computational mathematics and is usually not straightforward [2]. Further research is needed to develop and analyze efficient and robust numerical schemes. It is well-known that almost all algorithms used for the solution of deterministic ODEs may not work, or work very poorly for SDEs because they suffer from poor numerical convergence. Classical numerical schemes such as the Euler–Maruyama and Milstein methods correspond to truncating the stochastic Taylor expansion to generate global strong order 0.5 and order 1 schemes, respectively. Analogously to the deterministic case, stochastic Taylor schemes are obtained by truncating stochastic Taylor expansions. The practical difficulty of employing Taylor approximations is that they require the computation and evaluation of many derivatives. Other numerical schemes based on deterministic Runge–Kutta (RK) methods, which avoid the use of derivatives, have been derived and can be found in the textbook by Kloeden and Platen [2]. Similar to the ordinary case, the RK schemes are obtained by matching their truncated stochastic expansion about a point with the corresponding Taylor approximation. Since then, numerical solutions of SDEs have been investigated by many authors. In particular, many efficient numerical methods have been constructed for solving various types of SDEs with different properties; for example, see [2,1,5–7,3,8]. We also mention more recent theoretical results on the approximation of SDEs that deal with optimality and approximation under weaker and nonstandard assumptions on the drift and diffusion coefficients [6,9–13,7,14–19]. However, to the best of our knowledge, the DG finite element method has not been applied to SDEs. So a natural question is: is it possible to apply the standard DG method to SDEs of the form (1.1)? According to authors' knowledge, no DG finite element methods for (1.1) have been proposed in the literature, and this is the first time that the DG method has been applied to (1.1).