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A curve modeling method based on the envelope template

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ABSTRACT

This paper proposes a novel curve modeling method based on an envelope curve template. As the theoretical basis for the envelope template, a family of interpolating curves is generated by repeated interpolation of the given data points. On this basis, a family of one-parameter interpolating cylindrical surfaces is introduced, and then the envelope condition and its envelope surface are given, and the directrix of which serves as the prototype for curve modeling. After analyzing the geometric properties of the directrix of the envelope, the continuity conditions of two adjacent curve segments at the common point are derived. At a technical level, an envelope curve template is constructed, and the expected G^2 or C^2 composite curves can be obtained utilizing the envelope template sweeping over the data points.

1. Introduction

Curve modeling, an important basis in various modeling, has been mature and widely used in many fields. The representative methods such as Bezier curves [1,2], B-spline and NURBS curves [3–8], etc., have been extensively studied by Farin [9,10] and Hoschek [11]. It is found that they have some common characteristics: (a) the base functions are all parametric polynomials or polynomial splines, i.e. the algebraic curve serves as the prototype, simplifying the analytical calculation procedure, (b) the expected curve is continuously and smoothly formed by segments of curves, solving the oscillation problem caused by interpolation of high order curves, (c) the control points are introduced to adjust the shape of the curve as well as its continuity at the junction point. It is observed that, the envelope of a family of curves or surfaces plays an important role in the study of gear and cam mechanism, or the forming principle of NC machining [12]. Inspired by this observation, the authors proposed a surface modeling scheme based on an envelope template of a two-parameter family of interpolating surfaces [13]. As a companion piece to that article, this paper aims to explore the applications of the envelope theory in curve modeling.

It is indicated in differential geometry that the envelope of a family of plane curves is tangent at each of its points to a curve of the family, and all tangent points are characteristic points, the set of which is the envelope curve. Similarly, the envelope surface of a one-parameter family of surfaces is tangent to each surface of the family along a curve, and all tangent curves are characteristic curves, the set of which is the envelope surface. Still, the envelope of a two-parameter family of surfaces is tangent to each surface of the family at each of its points, and all tangent points are characteristic points, the set of which is the envelope surface. From these notes, it is concluded that the geometrical features of the family of curves or surfaces can be well reflected by the envelope. Meanwhile, the characteristic points are all salient points relative to the

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envelope curve or surface, controlling the oscillation effectively. Therefore, in curve or surface modeling, the envelope curve and surface can be served as geometric resources of high quality.

In this paper, the basis curves are first obtained by interpolating the original data points, and then the family of curves is generated by repeating interpolation of the points on these basis curves. Generally, the curves are space curves, while the envelope of space curves has not been studied or defined clearly. In order to solve this problem, this paper first introduces a family of cylindrical surfaces, and then gives the envelope of the one-parameter family of cylindrical surfaces, which is also a cylindrical surface. The directrix of the envelope originates from the space curves and possesses the characteristic of envelope curve, so it can be regarded as the envelope of the family of space curves. Based on the geometric properties of the directrix, the envelope template can be constructed for the composite curve with second order continuity.

2. The family of interpolating curves

Suppose given five data points expressed by an array as follows

$$\boldsymbol{P} = [\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \boldsymbol{P}_4, \boldsymbol{P}_5]^T . \tag{1}$$

Three subsets $\boldsymbol{P}_{0i}$ can be defined by three adjacent points in $\boldsymbol{P}$

$$\boldsymbol{P}_{0i} = [\boldsymbol{P}_i, \boldsymbol{P}_{i+1}, \boldsymbol{P}_{i+2}]^T , \quad i = 1, 2, 3. \tag{2}$$

By interpolating the three data points in $\boldsymbol{P}_{0i}$, three cubic interpolating curves $\boldsymbol{b}_i$ can be derived as follows

$$\begin{cases} \boldsymbol{b}_i = \lambda^T(u)\,\boldsymbol{P}_{0i}, \quad i = 1, 2, 3. \\ \lambda(u) = \begin{bmatrix} \alpha_1(u) & \alpha_2(u) & \alpha_3(u) \end{bmatrix}^T \\ \alpha_1(u) = 2\left(\dfrac{1}{2} - u\right)(1-u)^2, \qquad \alpha_2(u) = 4u(1-u), \qquad \alpha_3(u) = -2u^2\left(\dfrac{1}{2} - u\right), \qquad \displaystyle\sum_{k=1}^{3} \alpha_k = 1 \end{cases} \tag{3}$$

where $u \in [0, 1]$ is parameter of the interpolating curve, $\alpha_k(t)$ $(k = 1, 2, 3)$ denotes the basis function. The three interpolating curves are called basis curves; it is observed that two adjacent basis curves share two common data points, where the curve is only zero-order continuous. If one point from each curve with the same value of u is picked out, a new cubic interpolating curve can be obtained

$$\begin{cases} \boldsymbol{S} = \lambda^T(t)\,\boldsymbol{B}(u), \quad t, u \in [0, 1] \\ \boldsymbol{B}(u) = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3]^T . \end{cases} \tag{4}$$

Note that two independent variables exist in Eq. (4), i.e. t and u. On one hand, if u is fixed, a cubic interpolating curve will be described by variable t. On the other hand, if the parameter u varies between 0 and 1, a family of interpolating curves will be described along with the movement of the interpolating curve.

Substituting Eq. (3) into Eq. (4) leads to

$$\boldsymbol{S} = \lambda^T(t) \begin{bmatrix} \alpha_1(u)\boldsymbol{P}_1 + \alpha_2(u)\boldsymbol{P}_2 + \alpha_3(u)\boldsymbol{P}_3 \\ \alpha_1(u)\boldsymbol{P}_2 + \alpha_2(u)\boldsymbol{P}_3 + \alpha_3(u)\boldsymbol{P}_4 \\ \alpha_1(u)\boldsymbol{P}_3 + \alpha_2(u)\boldsymbol{P}_4 + \alpha_3(u)\boldsymbol{P}_5 \end{bmatrix}. \tag{5}$$

Considering the multiplication of two matrices, Eq. (4) can be represented by data points $\boldsymbol{P}$

$$\begin{cases} \boldsymbol{S} = \lambda^T(t)\,\Gamma^T(u)\,\boldsymbol{P} \\ \Gamma(u) = \begin{pmatrix} \alpha_1(u) & \alpha_2(u) & \alpha_3(u) & 0 & 0 \\ 0 & \alpha_1(u) & \alpha_2(u) & \alpha_3(u) & 0 \\ 0 & 0 & \alpha_1(u) & \alpha_2(u) & \alpha_3(u) \end{pmatrix}^T . \end{cases} \tag{6}$$

It is easy to verify that the matrix product in Eq. (6) satisfies the following equation

$$\Gamma(u)\lambda(t) = \Gamma(t)\lambda(u). \tag{7}$$

Therefore, Eq. (6) can be rewritten as

$$\boldsymbol{S} = (\Gamma(t)\,\lambda(u))^T\,\boldsymbol{P}. \tag{8}$$

Comparing Eqs. (6) and (8), it is found that the parameters u and t can be interchanged with each other, thus the property of mutual-exchange is possessed by the parameters.