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Stabilized and inexact adaptive methods for capturing internal layers in quasilinear PDE

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ABSTRACT

A method is developed within an adaptive framework to solve quasilinear diffusion problems with internal and possibly boundary layers starting from a coarse mesh. The solution process is assumed to start on a mesh where the problem is badly resolved, and approximation properties of the exact problem and its corresponding finite element solution do not hold. A sequence of stabilized and inexact partial solves allows the mesh to be refined to capture internal layers while an approximate solution is built eventually leading to an accurate approximation of both the problem and its solution. The innovations in the current work include a closed form definition for the numerical dissipation and inexact scaling parameters on each mesh refinement, as well as a convergence result for the residual of the discrete problem. Numerical experiments demonstrate the method on a range of problems featuring steep internal layers and high solution dependent frequencies of the diffusion coefficients.

1. Introduction

This investigation continues the work in [1,2] developing adaptive numerical methods for quasilinear partial differential equations featuring steep internal layers, in which the solution process starts on a coarse mesh where the problem is not yet resolved. Throughout the coarse mesh and preasymptotic regimes, standard methods such as Newton iterations are known to fail due to both the ill-conditioned and possibly indefinite Jacobians which are characteristic of the approximate discrete problems, and the partial resolution of the problem data. This paper specifies an appropriate set of parameters that may be used in the stabilized σ-Newmark strategy of [2] applied to quasilinear diffusion problems, where the layers develop from both the solution dependent coefficients and a variable dependent source. For the class of problems studied here

$$-\operatorname{div}(\kappa(u)\nabla u) - f(x) = 0, \quad u = 0 \text{ on } \partial\Omega, \quad \text{and} \tag{1.1}$$

$$-\operatorname{div}(\kappa(|\nabla u|^2)\nabla u) - f(x) = 0, \quad u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

in which the diffusion is bounded away from zero, local uniqueness of the solution is known, as well as approximation properties for the finite element solution using linear elements [3], assuming the mesh is sufficiently fine. For operators containing steep solution-dependent layers in the coefficients, the approximation properties of the discrete solution are useful only if the solution to the discrete nonlinear problem can be attained, and the current method attains such a solution by means of a sequence of approximate problems with inexact source functions that limit to the discrete problem.

The methodology is to first discretize (1.1) and (1.2) on each mesh refinement then linearize the resulting discrete problem, and to partially solve the sequence of resulting inaccurate and ill-conditioned coarse mesh problems by stabilized

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Newton-like iterations while adaptively refining the mesh leading to an accurate and efficient solve of an accurate discretization of the problem. In previous work the focus was on the stabilization of the Jacobian by a combination of regularization and added numerical dissipation. Currently, a formula for the numerical dissipation parameter is presented, along with a new inexact method designed for problems where the variable-dependent source dominates the residual of the Newton-like iterations.

Iterative rescaling techniques in the solution of nonlinear problems are not uncommon, see for instance the scaling iterative algorithm (SIA) of [4] for the solution of semilinear elliptic problems, in which the solution u is rescaled at each iteration. The rescaling of the Monge–Ampère equation in [5] to establish a fixed-point argument and recover a numerical solution without having to assume the solution is small enough motivated the current approach. Here, the inexact method rescales the variable dependent source until the solution iterates attain sufficient stability to solve for the given data.

Recent approaches such as [6,7] for monotone quasilinear problems use inexact linear and nonlinear solves to avoid over-solving for the residual when the Galerkin or discretization error is the dominant source of error. It is assumed in their analysis that the discrete problem on each refinement is well posed. In the problems studied here, the coarse mesh problem may not be well posed, and may be a sufficiently bad approximation of the exact problem that estimates of the different error sources are not necessarily well determined or useful. So long as the continuous problem is well-posed, at least locally in the neighborhood of a solution, the current algorithm could be designed to limit to the methodology of [6,7] as opposed to standard Newton iterations in the asymptotic regime.

In the current presentation the iterations on each preasymptotic refinement are ended when they stabilize to the predicted linear convergence rate, which is a function of the numerical dissipation parameter. Combined with the criteria that the residual from the linear solves on each mesh refinement must show sufficient decrease with respect to the residual on the previous refinement, the sequence of stabilized and inexact problems recovers the unscaled discrete problem on a mesh where it is better represented, and with an initial guess for the Newton-like iterations that yields the discrete problem solvable. This method predicts the stability of the solve and allows the sequence of coarse mesh problems to be solved approximately through the preasymptotic regime leading to an efficient solve in the asymptotic regime.

The remainder of the paper is organized as follows. Section 2 states the target problem class and the formulation of the discrete problems. Section 3 reviews the Jacobian stabilization techniques developed by the author in previous work, and which are further developed here. Section 4 presents a formulation for the numerical dissipation parameter γ and characterizes its properties within the adaptive framework; then Section 5 presents a formulation for the inexact scaling parameter δ and characterizes its convergence to unity within the adaptive method. Section 6 summarizes the results of the previous three sections into an adaptive algorithm and proves the convergence of the residual of the discrete problem. Finally, Section 7 demonstrates the method with a collection of numerical experiments featuring different types of internal layers.

The following notation is used throughout the rest of the paper. In defining the weak and bilinear forms in the next section $(u(x), v(x)) = \int_{\Omega} u(x)v(x)\,dx$, and in later sections the discrete inner product between vectors $u_k, v_k \in \mathbb{R}^n$ is denoted $\langle u_k, v_k\rangle$. The norm $\|\cdot\|$ where not otherwise specified is the L_2 norm. The nth iterate subordinate to the kth partition $\mathcal{T}_k$ is denoted u_k^n, while u^n is the nth iteration on a fixed partition and u_k is the final iteration on the kth mesh, taken as the approximate solution on $\mathcal{T}_k$.

2. Target problem class

The class of problems considered are quasilinear diffusion problems $F(u, x) = 0$, over polygonal domain $\Omega \subset \mathbb{R}^2$, with $F : X \times \Omega \to Y^*$ and $F'(u, x) := F_u(u, x) \in \mathcal{L}(X, Y^*)$, where $F(u, x)$ is given by

$$F(u, x) := -\operatorname{div}(\kappa(u)\nabla u) - f(x) = 0, \quad \text{in } \Omega \subset \mathbb{R}^2,\ u = 0 \text{ on } \partial\Omega, \quad \text{or} \tag{2.1}$$

$$F(u, x) := -\operatorname{div}(\kappa(|\nabla u|^2)\nabla u) - f(x) = 0, \quad \text{in } \Omega \subset \mathbb{R}^2,\ u = 0 \text{ on } \partial\Omega, \tag{2.2}$$

with $f(x) \in L_2(\Omega) \cap L_\infty(\Omega)$. Multiplication against test functions $v \in Y$ and integration by parts yields the weak form of each problem

$$B(u, v) = (\kappa(u)\nabla u, \nabla v) = (f, v), \quad \text{for all } v \in Y, \qquad \text{for (2.1)}, \tag{2.3}$$

$$B(u, v) = (\kappa(|\nabla u|^2)\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in Y, \qquad \text{for (2.2)}. \tag{2.4}$$

The linearized form induced by $F'(u, x) := F_u(u, x)$, is determined by taking the Gateaux derivative in direction $w \in X$ by $B'(u; w, v) = \lim_{t\to 0} d/dt\,(B(u + tw, v))$ yielding

$$B'(u; w, v) = (\kappa(u)\nabla w, \nabla v) + (\kappa'(u)w\nabla u, \nabla v), \qquad \text{for (2.1)}, \tag{2.5}$$

$$B'(u; w, v) = (\kappa(|\nabla u|^2)\nabla w, \nabla v) + (\kappa'(|\nabla u|^2)(2\nabla u \cdot \nabla w)\nabla u, \nabla v), \qquad \text{for (2.2)}. \tag{2.6}$$

Both types of problems fit into the context of [3] with the assumption that there is a solution $u \in H_0^1(\Omega) \cap W_{2+\varepsilon}^2(\Omega)$ and $F_u(u, x) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, in which case the solution u is an isolated solution, and approximation properties for the linear Lagrange finite element solution can be shown to hold, assuming the mesh size is fine enough.