Journal of Computational and Applied Mathematics 308 (2016) 301–317

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Generalized 2D Laguerre polynomials and their quaternionic extensions

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ARTICLE INFO

Article history:
Received 9 August 2015

MSC:
primary 81R30
46E22

Keywords:
Quaternion
Laguerre polynomials
Coherent states

ABSTRACT

A class of orthogonal polynomials in two quaternionic variables is introduced. This class serves as an analogous to the classical Zernike polynomials $Z_{m,n}^{(\beta)}(z,\overline{z})$ (arXiv:1502.07256, 2014). A number of interesting properties such as the orthogonality condition, recurrence relations, raising and lowering operators are discussed in detail. Particularly, the ladder operators, realized as differential operators in terms of the so-called Cullen derivatives, for these quaternionic polynomials are also studied. Some physically interesting summation and integral formulas are also proved, and their physical relevance briefly discussed.

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1. Introduction

Several mathematical properties of the 2D Hermite polynomials, introduced by Ito [1], were recently explored in [2,3]. These polynomials were used intensively to analyze several interesting physical problems, such as quantization [4–6], probability distributions [4], pseudo-bosons [4], and modular structures [7].

A quaternionic extension of 2D Hermite polynomials was introduced in [8] and used to obtain the coherent states and associated quaternionic regular and anti-regular subspaces in [9].

In [3], the authors developed a general scheme to obtain 2D polynomials. As an application of their approach, they developed the 2D analogue of Laguerre polynomials namely the Zernike (or disc) polynomials. Also, many interesting and useful mathematical properties of these polynomials were thoroughly studied and reviewed.

In the present work, we develop the quaternionic analogue of the general construction given in [3], and thereby obtained the quaternionic analogue of the 2D Laguerre polynomials. In this respect, we review in the next section some mathematical details on quaternions and the Cullen derivatives needed for our work. In Section 3, several properties are investigated and explored, particularly, the orthogonality relation. In Section 4, we consider the ladder operators of the complex and quaternionic Laguerre polynomials. In Sections 5 and 6, we investigate some summation and integral formulas. In Section 7, we briefly point out possible physical applications of the ladder operators, summation formulas and integral formulas developed earlier in Sections 4–6 along the lines of Refs. [4–6,9–11].

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http://dx.doi.org/10.1016/j.cam.2016.05.037
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2. Quaternions: a brief introduction

Let H denote the field of quaternions with elements of the form

$$\mathbf{q} = x_0 + x_1 i + x_2 j + x_3 k$$

where x_i, $i = 0, 1, 2, 3$, are real numbers, and i, j, k are imaginary units which satisfies

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i \quad \text{and} \quad ki = -ik = j.$$

The quaternionic conjugate of $\mathbf{q}$ is defined by

$$\overline{\mathbf{q}} = x_0 - x_1 i - x_2 j - x_3 k,$$

so that a real norm on H is defined by

$$|\mathbf{q}|^2 := \overline{\mathbf{q}}\,\mathbf{q} = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

for all $\mathbf{p}, \mathbf{q} \in H$. It is convenient in many applications to use the 2×2 matrix representation:

$$\mathbf{q} = x_0\,\sigma_0 + \underline{x} \cdot \underline{\hat{\sigma}}, \tag{2.1}$$

with $x_0 \in \mathbb{R}$, $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\sigma_0 = \mathbb{I}_2$, the 2×2 identity matrix and $\underline{\hat{\sigma}} = i(\sigma_1, -\sigma_2, \sigma_3)$, where the σ_ℓ, $\ell = 1, 2, 3$ are the usual Pauli matrices. Thus,

$$\mathbf{q} = \begin{pmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{pmatrix} \quad \text{and} \quad \overline{\mathbf{q}} = \mathbf{q}^\dagger \quad (\text{matrix adjoint}). \tag{2.2}$$

Introducing the polar coordinates:

$$x_0 = r\cos\theta, \qquad x_1 = r\sin\theta\sin\phi\cos\psi, \qquad x_2 = r\sin\theta\sin\phi\sin\psi, \qquad x_3 = r\sin\theta\cos\phi,$$

where $r \in [0, \infty)$, $\theta, \phi \in [0, \pi]$, and $\psi \in [0, 2\pi)$, we may write

$$\mathbf{q} = A(r)e^{i\theta\sigma(\widehat{n})}, \tag{2.3}$$

where

$$A(r) = r\sigma_0 \quad \text{and} \quad \sigma(\widehat{n}) = \begin{pmatrix} \cos\phi & e^{i\psi}\sin\phi \\ e^{-i\psi}\sin\phi & -\cos\phi \end{pmatrix}. \tag{2.4}$$

It is not difficult to show that the matrices $A(r)$ and $\sigma(\widehat{n})$ satisfy the conditions:

$$A(r) = A(r)^\dagger, \qquad \sigma(\widehat{n})^2 = \sigma_0, \qquad \sigma(\widehat{n})^\dagger = -\sigma(\widehat{n}), \qquad [A(r), \sigma(\widehat{n})] = 0. \tag{2.5}$$

Note

$$\mathbf{q}\,\mathbf{q}^{-1} = \mathbf{q}^{-1}\mathbf{q} = 1, \qquad \overline{\mathbf{pq}} = \overline{\mathbf{q}}\,\overline{\mathbf{p}}.$$

It is also well-known (see, e.g., [9]) that any $\mathbf{q} \in H$ can be also written in the 2×2 matrix representation as

$$\mathbf{q} = u_{\mathbf{q}}\,Z\,u_{\mathbf{q}}^\dagger, \tag{2.6}$$

where

$$u_{\mathbf{q}} = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \begin{pmatrix} \cos(\phi/2) & i\sin(\phi/2) \\ i\sin(\phi/2) & \cos(\phi/2) \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \in SU(2),$$

and $Z = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$, $z \in \mathbb{C}$. Let $d\omega(u_{\mathbf{q}})$ be the normalized Haar measure on the *compact* special unitary group $SU(2)$.

Definition 2.1 (*Cullen Derivative [12,13]*). Let Ω be a domain in H, and let $f : \Omega \longrightarrow H$ be a left regular function. For any $I \in \mathbb{S}$, the unit sphere of purely imaginary quaternions, and any point $\mathbf{q} = x + yI \in \Omega$ (x and y real numbers). The Cullen derivative $\partial_c f$ of f is defined as

$$\partial_c f(\mathbf{q}) = \begin{cases} \partial_I f(\mathbf{q}) := \dfrac{1}{2}\left(\dfrac{\partial f_I(x+Iy)}{\partial x} - I\dfrac{\partial f_I(x+Iy)}{\partial y}\right) & \text{if} \quad y \neq 0 \\ \dfrac{\partial f}{\partial x}(x) & \text{if} \quad \mathbf{q} = x \text{ is real.} \end{cases}$$

Similarly, for a right regular function f its Cullen derivative is defined as

$$\partial_c f(\mathbf{q}) = \begin{cases} \partial_I f(\mathbf{q}) := \dfrac{1}{2}\left(\dfrac{\partial f_I(x+Iy)}{\partial x} - \dfrac{\partial f_I(x+Iy)}{\partial y}I\right) & \text{if} \quad y \neq 0 \\ \dfrac{\partial f}{\partial x}(x) & \text{if} \quad \mathbf{q} = x \text{ is real.} \end{cases}$$

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