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Superconvergence in collocation methods for Volterra integral equations with vanishing delays[☆]

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ABSTRACT

In this paper, we investigate the optimal (global and local) convergence orders of the (iterated) collocation solutions for second-kind Volterra integral equations with vanishing delays on quasi-geometric meshes. It turns out that the classical global convergence results still hold under certain regularity conditions of the given functions. In particular, it is shown that the optimal local superconvergence order $p = 2m$ can be attained if collocation is at the m Gauss(–Legendre) points, which contrasts with collocations both on uniform meshes and on geometric meshes. Numerical experiments are performed to confirm our theoretical results.

1. Introduction

Consider the second kind Volterra integral equation (VIE) of the form

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [0, T], \tag{1.1}$$

where

$$(\mathcal{V}y)(t) := \int_0^t K_1(t, s)y(s)ds, \quad t \in I, \qquad (\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K_2(t, s)y(s)ds, \quad t \in I,$$

and $\theta(t) := qt$ $(0 < q < 1)$. The kernel functions $K_1(t, s)$ and $K_2(t, s)$ are supposed to be continuous in the domains $D := \{(t, s) : 0 \le s \le t \le T\}$ and $D_\theta := \{(t, s) : 0 \le s \le \theta(t), t \in I\}$, respectively.

Many basic mathematical models in population growth and relevant phenomena in biology are described by delay integral equations of the type

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in I, \tag{1.2}$$

with the Volterra integral operator

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^t K(t, s)y(s)ds, \quad t \in I,$$

or its nonlinear version (see, e.g., [1]). Eq. (1.2) may be viewed as a special case of (1.1) by setting $K_2 = -K_1 =: -K$.

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The existence, uniqueness and regularity properties of solutions to VIEs with vanishing delays have attracted a great deal of attention and are studied extensively (see, i.e., [2–4]). For classical (non-delay) Volterra integral equations, it is well known that the iterated collocation solution possesses the optimal superconvergence order $p = 2m$ at the mesh points of the underlying uniform mesh. This property is well preserved for VIEs with non-vanishing delays (e.g., constant delays) on suitable constrained mesh, and a detailed discussion can be found in [5,6]. Unfortunately, these superconvergence properties on uniform mesh cannot carry over to VIEs with vanishing delays. In fact, the works of Brunner [7] and Takama et al. [8] showed that for Eq. (1.1), the optimal local convergence order appears to be $p = 2m - 1$ at most. In 2001, Brunner et al. [9] proved that an order $p = 2m - \varepsilon_N$ can be attained on suitable geometric meshes, where $\varepsilon_N \to 0$ as the mesh points $N \to \infty$ (see also [10]). In addition, a complete analysis on the attainable order for (1.1) on uniform meshes was given by Brunner et al. [11]. The authors showed that the local superconvergence order of the iterated collocation solution to (1.1) cannot exceed $p = m + 2$, and this order can only be obtained when $q = 1/2$ and m is even (see also [12]).

In the present paper, our aim is to study whether the optimal superconvergence order $p = 2m$ is attainable for the iterated collocation solution to (1.1) on quasi-geometric meshes. The quasi-geometric mesh was first introduced by Bellen et al. [13] for the purpose of numerical stability. In 2002, Bellen [14] proposed a continuous Runge–Kutta method to approximate the solution of proportional delay differential equations on quasi-geometric meshes, and proved that the scheme may preserve the optimal order. For Volterra integral–differential equations (VIDEs) with proportional vanishing delays, Brunner [15] proved that the collocation solution can attain optimal local superconvergence on quasi-geometric meshes. In 2006, Bellen et al. [16] generalised the convergence results in [15] to the case of nonlinear vanishing delay. Concerning VIEs with vanishing delays, however, the analysis of the optimal convergence order of collocation solutions on quasi-geometric meshes is still an open problem (see [17]), and this will be the focus of our present paper.

The outline of this paper is as follows. In Section 2, (iterated) collocation solutions and the corresponding fully discretised collocation solutions to (1.1) are introduced, and the existence and uniqueness of the collocation solutions are also discussed. In Section 3, the global convergence orders of the (iterated) collocation solutions are investigated. In Section 4, the local convergence analysis is carried out and the optimal superconvergence order $p = 2m$ on quasi-geometric meshes is obtained. In Section 5, some numerical experiments are presented to illustrate the theoretical results. Finally, some concluding remarks are given in Section 6.

2. Collocation methods on quasi-geometric meshes

In this section, we will establish four kinds of collocation solutions to (1.1) on quasi-geometric meshes, namely, a direct collocation solution $u_h(t)$, an iterated collocation solution $u_h^{it}(t)$, and the corresponding fully discretised collocation solutions $\hat{u}_h(t)$ and $\hat{u}_h^{it}(t)$. Moreover, the existence and uniqueness of the collocation solutions are also discussed.

2.1. Collocation solution in $S_{m-1}^{(-1)}(\tilde{I}_h)$

We first assume that for certain $t_0 > 0$, an approximate value $y_0(t)$ to the exact solution $y(t)$ of (1.1) on $[0, t_0]$ satisfies

$$\|y - y_0\|_{0,\infty} := \max_{t\in[0,t_0]} |y(t) - y_0(t)| \le \tilde{C} t_0^{p_0} \tag{2.1}$$

for some $p_0 \ge 1,\ p_0 \in \mathbb{N}$. Here, the constant $\tilde{C}$ does not depend on t_0 or p_0. This approximation can be generated either by resorting to proper Taylor series for $y(t)$, or by utilising appropriate collocation methods. For the given value $y_0(t)$, we consider the following problem

$$\begin{cases} z(t) = g(t) + (\mathcal{V}z)(t) + (\mathcal{V}_\theta z)(t), & t \in \tilde{I} := (t_0, T], \\ z(t) = y_0(t), & t \in [0, t_0]. \end{cases} \tag{2.2}$$

Note that Eq. (2.2) is equivalent to (1.1) if $y_0(t)$ coincides with the exact solution $y(t)$ for $t \in [0, t_0]$. Let

$$t_0 = q^M T \quad \text{for some } M \in \mathbb{N}.$$

Then, we define the set of macro-mesh points

$$\Omega_1 := \{\xi_s : t_0 = \xi_0 < \xi_1 < \cdots < \xi_M = T,\ \xi_s := q^{M-s} T\ (0 \le s \le M)\},$$

and the macro-interval $I^{(s)} := (\xi_s, \xi_{s+1}]$ can be divided as

$$I_h^{(s)} := \{t_n^{(s)} : \xi_s = t_0^{(s)} < t_1^{(s)} < \cdots < t_N^{(s)} = \xi_{s+1}\}.$$

For linear delay function $\theta(t) = qt$, we often choose the uniform division, which implies that the meshes $\tilde{I}_h := \bigcup_{s=0}^{M-1} I_h^{(s)}$ are not only constrained but also θ-invariant, i.e.,

$$\theta(I_h^{(s+1)}) = I_h^{(s)}, \quad s = 0, 1, \ldots, M-1.$$