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A finite-difference method for a singularly perturbed delay integro-differential equation

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ABSTRACT

We consider the singularly perturbed initial value problem for a linear first order Volterra integro-differential equation with delay. Our purpose is to construct and analyse a numerical method with uniform convergence in the perturbation parameter. The numerical solution of this problem is discretized using implicit difference rules for differential part and the composite numerical quadrature rules for integral part. On a layer-adapted mesh error estimations for the approximate solution are established. Numerical examples supporting the theory are presented.

1. Introduction

In this paper, we consider the following singularly perturbed delay Volterra integro-differential equation in the interval $\bar{I} = [0, T]$:

$$\begin{aligned} Lu &:= \varepsilon u'(t) + a(t)u(t) + b(t)u(t-r) + \int_0^t \{K(t,s)u(s) + L(t,s)u(s-r)\}ds \\ &= f(t), \quad t \in I, \end{aligned} \tag{1.1}$$

$$u(t) = \psi(t), \quad t \in I_0, \tag{1.2}$$

where $I = (0, T] = \cup_{p=1}^{m} I_p, I_p = \left\{t : r_{p-1} < t \le r_p\right\}, 1 \le p \le m$ and $r_s = sr$, for $0 \le s \le m$ and $I_0 = [-r, 0]$.

$\varepsilon \in (0, 1]$ is the perturbation parameter and r is a constant delay, which is independent of ε. We assume that $a(t) \ge \alpha > 0,\ b(t),\ f(t)\ (t \in \bar{I}),\ \psi(t)\ (t \in I_0), K(t,s)$ and $L(t,s)$ $((t,s) \in \bar{I} \times \bar{I})$ are given sufficiently smooth functions satisfying certain regularity conditions to be specified. The initial value problem (1.1) has in general boundary layers on the right side of each point $t = r_s$ $(0 \le s \le m-1)$ for small values of ε (see Section 2).

Volterra delay-integro-differential equations (VDIDE's) arise widely in scientific fields such as biology, ecology, medicine and physics [1–5]. This class of equations plays an important role in modelling diverse problems of engineering and natural science, and hence has led researchers to develop a theory and numerical analysis for VDIDE's.

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Differential equations with a small parameter ε multiplying the highest order derivative terms are said to be singularly perturbed and normally boundary layers occur in their solutions. These equations play an important role in today's advanced scientific computations. Many mathematical models starting from fluid dynamics to the problems in mathematical biology are modelled by singularly perturbed problems. Typical examples include high Reynold's number flow in the fluid dynamics, heat transport problem, etc. For more details on singular perturbation, one can refer the books [6–10] and the references therein. It is well known that, for small values of ε, standard numerical methods for solving such problems are unstable and do not give accurate results. Therefore, it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the parameter value ε, i.e., methods that are convergent ε-uniformly. These include fitted finite difference methods, finite element methods using special elements such as exponential elements, and methods which use a priori refined or special non-uniform grids which condense in the boundary layers in a special manner. The various approaches to the design and analysis of appropriate numerical methods for singularly perturbed differential equations can be found in [11–13,7,14,8,15,16] (see also references cited in them). One of the simplest ways to derive parameter uniform methods consists of using a class of special piecewise uniform meshes (a Shishkin mesh, see e.g. [7,16]), which are constructed a priori and depend on the perturbation parameter, the problem data, and the number of corresponding mesh points.

For a survey of early results in the theoretical analysis of singularly perturbed Volterra integro-differential equations (VIDE's) and in the numerical analysis and implementation of various techniques for these problems we refer to the book [17]. Various approximating aspects for singularly perturbed VIDE's have also been investigated in [12,18–24].

Recently, there has been a growing interest in the numerical solution of VDIDE's. For example, Koto [25] studied stability of Runge–Kutta method for VDIDE's with a constant delay. The qualitative behaviour of numerical approximations to a nonlinear VDIDE with unbounded delay is investigated in [26]. Zang and Vandewalle [27] gave a numerical method based on the combination of the general linear methods with compound quadrature rules for VDIDE's. Gan [28] studied the analytic and numerical dissipativity of θ-methods for nonlinear VDIDE's. The adaptation of linear multistep methods for VDIDE's has been discussed in [29]. Shakourifar and Enright [30] considered standard software based on the collocation method for solving VDIDE's. The numerical stability of linear VDIDE's with real coefficients has been discussed by Zhao et al. [31]. Belloura and Bousselsal [32] used the Taylor polynomial method for approximating VDIDE's. Zhao et al. [33] constructed a methodology based on the sinc collocation technique to approximate pantograph VDIDE's.

The above mentioned papers, related with VDIDE's were only concerned with the regular cases. Also singularly perturbed Volterra delay-integro-differential equations (SPVDIDE's) frequently arise in many scientific applications. Wu and Gan [34] investigated error behaviour of linear multistep methods applied to SPVDIDE's and derived global error estimates $A(\alpha)$-stable linear multistep methods with convergent quadrature rule. He and Xu [35] discussed the exponential stability of impulsive SPVDIDE's. Amiraliyev and Yilmaz [13] gave an exponentially fitted difference method on a uniform mesh for (1.1)–(1.2) except for a delay term in differential part and shown that the method is first-order convergent uniformly in ε.

The aim of this paper is discretizing (1.1)–(1.2) using a numerical method, which is composed of an implicit finite difference scheme on piecewise uniform Shishkin-meshes on each time-subinterval. The scheme is constructed by the method based on using appropriate quadrature rules with the weight and remainder terms in integral form. This method of approximation has the advantage that the schemes can be effectively applied also in the case when the original problem has a solution with certain singularities (presence of boundary layer, nonsmooth solutions, etc.). But we need to have a prior information about the location and width of the layers to generate the mesh. It is proved under some additional conditions that our scheme is stable uniformly with respect to ε and that its solution approximates the solution of (1.1)–(1.2) with the error $O(N^{-1}\ln N)$, where N is the mesh parameter. The approach to construct discrete problem and error analysis for approximate solution is similar to those one's from [11–13].

The structure of this paper is as follows. In Section 2, we state some important properties of the exact solution. In Section 3, we describe the finite difference discretization and introduce the piecewise uniform grid. In Section 4, we present the error analysis for the approximate solution. Uniform convergence is proved in the discrete maximum norm. Numerical results are given in Section 5 to support the predicted theory. The paper ends with a summary of the main conclusions.

Notation. Throughout the paper, C denotes a generic positive constant independent of ε and the mesh parameter. Some specific, fixed constants of this kind are indicated by subscripting C. For any continuous function $g(t)$, $\|g\|_\infty$ denotes a continuous maximum norm on the corresponding closed interval, in particular we shall use $\|g\|_{\infty,\bar{I_p}} = \|g\|_{\infty,p} = \max_{t\in\bar{I_p}}|g(t)|$, $0 \le p \le m$. We also use $I_p^* = \{t : 0 < t \le r_p\}$, $1 \le p \le m$.

2. Asymptotic behaviour of the exact solution

In this section, we give a priori estimates for the solution of the problem (1.1)–(1.2), which indicate the asymptotic behaviour of the solution and its first derivative in respect to perturbation parameter. These estimates are unimprovable in terms of the view of behaviour in ε and will be used in order to analyse the numerical solution.