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Truncated generalized averaged Gauss quadrature rules

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ABSTRACT

Generalized averaged Gaussian quadrature formulas may yield higher accuracy than Gauss quadrature formulas that use the same moment information. This makes them attractive to use when moments or modified moments are cumbersome to evaluate. However, generalized averaged Gaussian quadrature formulas may have nodes outside the convex hull of the support of the measure defining the associated Gauss rules. It may therefore not be possible to use generalized averaged Gaussian quadrature formulas with integrands that only are defined on the convex hull of the support of the measure. Generalized averaged Gaussian quadrature formulas are determined by symmetric tridiagonal matrices. This paper investigates whether removing some of the last rows and columns of these matrices gives quadrature rules whose nodes live in the convex hull of the support of the measure.

1. Introduction

Let $d\sigma$ be a nonnegative measure with infinitely many points of support. The smallest closed interval that contains the support of $d\sigma$ is denoted by $[a, b]$ with $-\infty \leq a < b \leq \infty$, and we assume that the distribution function σ has infinitely many points of increase in this interval. If σ is an absolutely continuous function, then $d\sigma(x) = w(x)\,dx$ on $\mathrm{supp}(d\sigma)$, where $w(x) \geq 0$ is a weight function. Let $\mathbb{P}_k$ denote the set of all polynomials of degree at most k and introduce the quadrature formula (abbreviated q.f.)

$$Q_n[f] = \sum_{j=1}^{n} \omega_j f(x_j)$$

with real distinct nodes $x_1 < x_2 < \cdots < x_n$ and real weights ω_j. We say that Q_n is a $(2n-m-1, n, d\sigma)$ q.f. if the remainder term $R_n[f]$, defined by

$$\int f(x)\,d\sigma(x) = Q_n[f] + R_n[f],$$

satisfies $R_n[f] = 0$ for all $f \in \mathbb{P}_{2n-m-1}$. The rule Q_n then is said to have *algebraic degree of precision* $2n-m-1$. Here m is an integer such that $0 \leq m \leq n$. If in addition all quadrature weights ω_j are positive, then Q_n is said to be a *positive* $(2n-m-1, n, d\sigma)$ q.f. Furthermore, we say that a polynomial $t_n = \prod_{j=1}^{n}(x-x_j)$ generates a $(2n-m-1, n, d\sigma)$ q.f. if its zeros x_j are real and simple, and the q.f. with nodes $x_1, x_2, \ldots, x_n$ is a $(2n-m-1, n, d\sigma)$ q.f. A $(2n-m-1, n, d\sigma)$ q.f. is *internal* if all its nodes are in the closed interval $[a, b]$. A node not belonging to the interval $[a, b]$ is said to be *external*.

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It is well known that an ℓ-node Gauss quadrature rule associated with the measure $d\sigma$ can be represented by an $\ell \times \ell$ real symmetric tridiagonal matrix $J_\ell^G(d\sigma)$ determined by the recursion coefficients of the first ℓ orthogonal polynomials associated with the measure $d\sigma$; see, e.g., Gautschi [1] or below. Spalević [2] proposed that the leading $(\ell-1)\times(\ell-1)$ tridiagonal submatrix of $J_\ell^G(d\sigma)$ be flipped right–left and upside-down, and appended to $J_\ell^G(d\sigma)$ to obtain a new symmetric tridiagonal matrix $J_{2\ell-1,\ell-1}$ of order $2\ell-1$. The latter matrix defines a $(2\ell-1)$-node quadrature formula referred to as a *generalized averaged Gaussian quadrature formula*. Spalević showed in [3] that these quadrature rules may yield a smaller quadrature error than what can be explained by just considering their algebraic degree of precision. This makes the generalized averaged Gaussian quadrature formulas attractive to use when it is inexpensive to evaluate the integrand at the nodes, but it is expensive or cumbersome to compute the moment information needed to determine the Gauss rule. Applications of generalized averaged Gaussian quadrature rules to problems of this kind are described in [4], where the quadrature rules are used to estimate quantities of interest in network analysis. In this application, the computation of each row and column of the matrix $J_\ell^G(d\sigma)$ requires the evaluation of a matrix–vector product with the adjacency matrix that defines the graph. The evaluation of matrix–vector products is expensive when the adjacency matrix is large. Gautschi describes in [1, Section 2.2], as well as in [5], other applications with measures $d\sigma$, for which the recursion coefficients for the associated orthogonal polynomials are not explicitly known and therefore have to be computed in order to determine Gaussian quadrature formulas. Gautschi proposed to compute approximations of the recursion coefficients by discretizing the measure $d\sigma$ and applying a Stieltjes procedure using the approximations of the required inner products determined by the discretized measure. These computations may be cumbersome if a fine discretization is required and a Gauss rule of high order is desired. It may then be attractive to use generalized averaged Gaussian quadrature formulas instead of standard Gauss rules, because the former often give higher accuracy when the same recursion coefficients are available for their construction; see Section 5 for computed examples.

It is the purpose of the present paper to describe extensions of the generalized averaged Gaussian quadrature formulas introduced in [2]. Section 2 discusses the extension of the real symmetric tridiagonal $\ell \times \ell$ matrix $J_\ell^G(d\sigma)$ associated with an ℓ-node Gauss quadrature rule with respect to the measure $d\sigma$ to a real symmetric tridiagonal matrix $J_{k+\ell,k}$ of order $k+\ell$ by appending a fairly arbitrary real symmetric tridiagonal matrix of order k to $J_\ell^G(d\sigma)$. Similarly as the generalized averaged Gaussian formulas introduced by Spalević [2], these extensions may yield a smaller quadrature error than the underlying ℓ-node Gaussian quadrature formula. Section 3 is concerned with the possible presence of exterior nodes of generalized averaged Gaussian quadrature formulas. It is well known that the nodes of (standard) Gaussian quadrature formulas live in the convex hull of the support of the measure that determines the formulas. Spalević showed that the generalized averaged Gaussian quadrature formulas in [2] may have one node to the right or to the left of the convex hull of the support of the measure. It therefore may not be possible to apply these quadrature rules when the integrand is defined on the convex hull of the support of the measure only. To remedy this shortcoming, truncated generalized averaged Gaussian quadrature rules were introduced in [4]. These rules are obtained by removing the last few rows and columns of the real symmetric tridiagonal matrix $J_{2\ell-1,\ell-1}$ associated with the generalized averaged Gaussian quadrature rules described in [2]. These truncated generalized averaged Gaussian quadrature rules have the same algebraic degree of precision as the non-truncated ones. We investigate these rules by using results by Peherstorfer [6] on positive quadrature rules. Section 4 presents a detailed analysis of truncated generalized averaged Gaussian quadrature rules obtained by appending only one row and column to the matrix $J_\ell^G(d\sigma)$, and investigates for classical measures $d\sigma$ when these rules are internal. Section 5 presents a few computed examples and Section 6 contains concluding remarks.

2. Generalized averaged Gaussian quadrature formulas

The following result by Peherstorfer [6, Lemma 1.1] is important for the investigation of generalized averaged Gaussian quadrature rules. The lemma uses properties of so-called associated polynomials. These polynomials are defined below.

Lemma 2.1. *Let $n, m \in \mathbb{N}_0$. Then $t_n \in \mathbb{P}_n$ determines a positive $(2n-1-m, n, d\sigma)$ q.f. if and only if t_n is orthogonal to $\mathbb{P}_{n-m-1}$ with respect to $d\sigma$, t_n has n simple zeros in the open interval (a, b), and the zeros of t_n and $t_{n-1}^{(1)}$ interlace, where $t_{n-1}^{(1)}$ denotes the associated polynomial to t_n.*

Let p_k denote the monic polynomial of degree k that is orthogonal to $\mathbb{P}_{k-1}$ with respect to $d\sigma$, i.e.,

$$\int_a^b x^j p_k(x) d\sigma(x) = 0, \quad j = 0, 1, \ldots, k-1.$$

Recall that the polynomials $\{p_k\}_{k=0}^{\infty}$ satisfy a three-term recurrence relation of the form

$$p_{k+1}(x) = (x-\alpha_k)p_k(x) - \beta_k p_{k-1}(x), \quad k = 0, 1, \ldots, \tag{2.1}$$

where $p_{-1}(x) \equiv 0$, $p_0(x) \equiv 1$, $\alpha_k \in \mathbb{R}$, and $\beta_k > 0$ for all k; see, e.g., Gautschi [1] for details. The ℓ-node Gaussian rule

$$Q_\ell^G[f] = \sum_{j=1}^{\ell} \omega_j^G f\left(x_j^G\right) \tag{2.2}$$