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Finite difference methods for pricing American put option with rationality parameter: Numerical analysis and computing

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ABSTRACT

In this paper finite difference methods for pricing American option with rationality parameter are proposed. The irrational exercise policy arising in American options is characterized in terms of a rationality parameter. The model is formulated in terms of a new nonlinear Black–Scholes equation that requires specific numerical methods. Although the solution converges to the solution of the classical American option price when the parameter tends to infinity, for finite values of the parameter the classical boundary conditions cannot apply and we propose specific ones. A logarithmic transformation is used to improve properties of the numerical solution that is constructed by explicit finite difference method. Numerical analysis provides stability conditions for the methods and its positivity. Properties of intensity function are studied from the point of view of numerical solution. Concerning the numerical methods for the original problem we propose the θ-method for time discretization, thus including explicit, fully implicit and Crank–Nicolson schemes as particular cases. The nonlinear term is treated by a Newton method. The convergence rate is illustrated by numerical examples.

1. Introduction

It is well known from the basics of financial options that an American option gives the right to the owner to exercise it and receive the corresponding payoff. Irrational behaviour is frequent in market trading due to many reasons such as emotional reactions or imperfect information [1–3]. In the case that rational exercise takes place, dynamic hedging methodology allows to pose the pricing problem as a linear complementarity problem associated to the linear Black–Scholes equation, which can be equivalently formulated as an optimal stopping time problem [4]. However, empirical studies illustrate a lot of situations when irrational exercise takes place. For example, in [1] the irrational exercise of S & P 100 call and put options is illustrated, while [2] shows that the clients of discount brokers irrationally exercise their calls too early, mainly in the case of less sophisticated investors. Sometimes, the irrational exercise related rely on specific circumstances around the global investment position, for example when the American option is part of a hedging strategy in which the optimal exercise rule results to be not optimal.

Recently, in [3] a new nonlinear Black–Scholes model that takes into account irrational exercise behaviour is proposed. More precisely, the authors characterize the rationality of exercise of an American put option in terms of a rationality

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parameter by means of an intensity based model for the valuation of the American puts, so that the exercise decision occurs as the first jump time of a process with stochastic intensity. Thus, the exercise intensity depends on how profitable is to exercise, i.e. the larger the difference between the payoff and the value of the American put if not exercised the greater the exercise intensity. Thus, the rationality parameter accounts for the dependence of the exercise intensity in terms of the profitability. In [3], the authors provide a probabilistic proof of the existence of solution as well as of the convergence of the model to the solution of the American put option in the rational case when the rational parameter tends to infinity. Moreover, by using Feynman–Kac theorem, the associated nonlinear Black–Scholes model is posed.

In the present paper, we confirm numerically that the solution of the irrational problem proposed in [3] for large values of rationality parameter tends to the solution of the rational American option problem.

We address the numerical approach of the solution of the nonlinear model for a vanilla American put proposed in [3]. In Section 2 we summarize and adapt the irrational model for the numerical treatment. We introduce a new boundary condition when the price of the underlying asset tends to zero. Note that the classical boundary condition for zero underlying price of the rational American option does not apply in the irrational case. Appropriate boundary conditions are specially required for the localization procedure to confine the problem in a bounded domain as previous step to the use of numerical techniques. Apart from the intensity functions proposed in [3], we introduce two additional smooth intensity functions. In Section 3 we propose a numerical solution based on a θ-method for PDE discretization combined with a Newton method for the nonlinear term. In Section 4, a transformation technique is used in order to take benefit of some numerical advantages. In fact, the PDE problem is transformed into a problem with constant coefficients in diffusion, convection and reaction parts of the equation together with a nonlinear term involving the intensity function. The transformed equation is discretized with an explicit in time finite difference method. Positivity, stability and consistency of the proposed explicit method are studied in Section 5.

After addressing the stability analysis, some numerical examples in Section 6 illustrate the expected order of convergence for the classical explicit, fully implicit and Crank–Nicolson particular cases of the θ-method as well as the proposed explicit method.

2. Irrational exercise model for American options in the single asset case

As departure point we assume the standard conditions of Black–Scholes model and consider a risky asset, the price of which follows a geometric Brownian motion under risk neutral probability, so that the price at time t, S_t, satisfies the following stochastic differential equation

$$dS_t = r\,S_t\,dt + \sigma\,dW_t, \quad S_0 \text{ given}, \tag{1}$$

where r denotes the constant risk-free interest rate, σ is the constant instantaneous volatility of the asset and W_t represents a standard Brownian motion. If we consider an American put option with strike price E and maturity T on the previous asset, the exercise value at time $t < T$ is given by $(E - S_t)^+$, so that the arbitrage-free value of the American put, $P_t^A = P^A(t, S_t)$, can be characterized as the solution of an optimal stopping time problem

$$P_t^A = P^A(t, S_t) = \sup_{t\le\tau\le T} E_t\left[\exp(-r(\tau - t))(E - S_\tau)^+\right], \tag{2}$$

where E_t denotes the conditional expectation to time t in the risk-neutral probability measure. Thus, there exists an optimal stopping time, τ^*, for which the supremum is attained.

In [3], in order to model irrational exercise, the authors introduce the irrational exercise rule, τ, as the minimum of the terminal time, T, and the first jump time of a point process with stochastic intensity $(\mu_t)_{0\le t\le T}$ (see [5], for example). Next, in terms of this family of intensity functions, a strictly positive rationality parameter, λ, measuring the rationality of the behaviour of the American option owner, is introduced. Moreover, if λ is the parameter of the family of intensity functions, f^λ, and we denote by τ^λ the associated exercise strategies, when λ tends to infinity we recover the arbitrage free price of the American put (i.e. the one associated to a fully rational exercise). If the exercise policy depends on how profitable is to exercise, then the relation between profitability and stochastic exercise intensity can be written in the form:

$$\mu_t = f((E - S_t) - P(t, S_t; \tau)), \tag{3}$$

where $f : [-K, K] \to [0, +\infty)$ denotes the intensity function and τ is the exercise strategy. In Theorem 2 in [3], sufficient conditions for an index of a family of intensity functions to be a rationality parameter are stated. Hereafter we recall the theorem.

Theorem 2.1. *Let $(f^\lambda)_{\lambda>0}$ be a family of positive deterministic intensity functions. For each $\lambda > 0$, let the stochastic intensity process be given by*

$$\mu_t^\lambda = f^\lambda((E - S_t)^+ - P^\lambda(t, S_t)),$$

where $P^\lambda(t, S_t) = P^\lambda(t, S_t; \tau^\lambda)$ and τ^λ is the exercise strategy of the American put given as the first jump time of a point process with intensity μ^λ. Let $\nu_\lambda(x) = 1_{(x<0)} \sup_{y\le x} f^\lambda(y) + 1_{(x\ge0)} \sup_{y\ge x} f^\lambda(y)$ and assume that

- *$\nu_\lambda(0+) \to \infty$ as $\lambda \to \infty$.*
- *There exists a function $\epsilon : (0, \infty) \to (0, \infty)$ such that $\nu_\lambda(-\epsilon(\lambda)) \to 0$ and $\epsilon(\lambda)\nu_\lambda(0-) \to 0$ as $\lambda \to \infty$.*