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Explicit, parallel Poisson integration of point vortices on the sphere

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ABSTRACT

Point vortex models are frequently encountered in conceptual studies in geophysical fluid dynamics, but also in practical applications, for instance, in aeronautics. In spherical geometry, the motion of vortex centres is governed by a dynamical system with a known Poisson structure. We construct Poisson integration methods for these dynamics by splitting the Hamiltonian into its constituent vortex pair terms. From backward error analysis, the method is formally known to provide solutions to a modified Poisson system with the correct bracket, but with a modified Hamiltonian function. Different orderings of the pairwise interactions are considered and also used for the construction of higher order methods. The energy and momentum conservation of the splitting schemes is demonstrated for several test cases. For particular orderings of the pairwise interactions, the schemes allow scalable parallelization. This results in a linear – as opposed to quadratic – scaling of computation time with system size when scaling the number of processors accordingly.

1. Motivation

A point vortex represents a singular measure solution to the vorticity equation for two-dimensional, incompressible fluid flow. A point vortex model consists of multiple point vortices mutually interacting. The motion of each point vortex is dictated by the flow field induced by the other vortices and by external forcing, e.g. topography. Point vortices were introduced by Helmholtz [1] and have since been the subject of much study; see for example [2–4].

Dynamical studies of point vortex systems provide insight into the (qualitative) behaviour of fluid dynamics. The series of papers by Newton et al. [4–7] discuss relative equilibria and the conditions for integrability of the dynamics. Vortex dynamics were studied extensively by Aref who compiled an extensive review on their history [8]. Newton [9] discusses the future of point vortex research in the "post-Aref era".

In statistical fluid mechanics, the behaviour of point vortex systems has been studied as a model for two-dimensional turbulence in the limit of an infinite number of vortices. This was first done by Onsager [10], who provided an explanation for the formation of clusters of like-signed vortices in a bounded domain. This research has since been continued by, amongst others, Joyce and Montgomery [11,12], Pointin and Lundgren [13], Eyink and Spohn [14], and Lions and Majda [15]. Such results are of interest in the fields of geophysical fluid dynamics [16] and stellar dynamics [17]. Some of Onsager's statements were tested numerically by Bühler [18].

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Point vortices and their three-dimensional generalization, vortex filament methods, are also used as a discretization of practical fluid flows in engineering applications [19]. By using a large number of point vortices a continuous velocity field is approximated. Such techniques find practical application in the works of Chatelain et al. [20], Rossinelli et al. [21]. Winckelmans et al. [22] and Rossinelli and Koumoutsakos [23] present the fast multipole, vortex-in-cell and hybrid methods that are used for computing these large systems. Sakajo [24] has developed a fast tree-code reducing the $\mathcal{O}(N^2)$ computational cost to $\mathcal{O}(N \log^3 N)$ for N interacting point vortices. Regularized approximations to the delta distributions provide more accurate representations of continuous vorticity fields, but their solutions are no longer exact, as the kernel itself ought to deform due to shearing [25,26].

It is important to develop efficient time integrators for point vortex methods for two reasons. First, the use of very large numbers of point vortices, as required for accurate approximation of continuous fluids, is hampered by the quadratic complexity of the pairwise coupling between vortices, i.e. evaluations of the vector field with N vortices requires N^2 operations. Second, the concept of numerical stability of a system of point vortices on planar geometry is not without ambiguity. Equilibria only exist for certain configurations, and are never asymptotically stable since the dynamics are Hamiltonian. The simplest nontrivial system is a pair of like-signed vortices, whose solution is periodic. If a contracting method such as backward Euler is employed, the vortices will eventually approach one another, and the derivatives grow unbounded. If an expanding method such as forward Euler is employed, the vortices will drift apart and the trajectories will grow without bounded. Hence, even for this simple configuration some degree of energy conservation is necessary to maintain a bounded solution with bounded derivative.

Recently, Vankerschaver and Leok [26] have developed a Poisson integrator for point vortex systems via the construction of a higher dimensional linear Lagrangian. The associated dynamics project down onto solutions of the point vortex equations on the sphere. The resulting integrator exactly conserves the Casimirs and momentum of the point vortex dynamics and also has good conservation of energy. The implicit definition, however, requires the use of an iterative solver.

We give an interpretation of the point vortex method in light of the approach first communicated by McLachlan [27] for discretizing Hamiltonian PDEs; namely as a scheme that discretizes the Poisson structure and Hamiltonian separately. With a vorticity field given as a sum of point vortices, the quadrature of the Hamiltonian functional is evaluated exactly as a sum of pointwise values. We do not consider regularizations of the vortices, but they could be accommodated in the quadrature scheme for the Hamiltonian. The Poisson bracket is discretized exactly for a particular class of functionals.

A numerical integrator for these dynamics follows from splitting the Hamiltonian into its constituent pairwise terms. The scheme developed is Poisson, explicit and allows scalable parallelization. It may also be applied to regularized point vortices, provided the kernel is rotation- and translation-invariant. The method requires an explicit expression for the pairwise flow map for the two-vortex system. Any regularization that maintains a pairwise Hamiltonian form will have three Poisson-commuting first integrals and is thus integrable. Both rotation of the sphere and topography introduce only decoupled, splittable terms in the Hamiltonian.

The remainder of this paper is organized as follows. Section 2 describes two-dimensional incompressible fluid flow in Hamiltonian form. Section 3 discusses the discretization according the ideas of McLachlan [27]. A Poisson integrator for the resulting point vortex description for fluids is developed in Section 4. The parallelization of this method is discussed in Section 5. Numerical results and comparisons of computation times are presented in Sections 4 and 5, respectively. Finally, in Section 6 we state conclusions and discuss the extension of the method to practical applications.

2. Continuous Hamiltonian description

The barotropic quasi-geostrophic equations on the unit sphere provide a simple model for studying geophysical fluid dynamics [28]. Point vortex representations capture much of the system's dynamics, for instance the formation of coherent vortical structures over long time [10]. This is a consequence of the existence of negative temperature states, that are possible due to the bounded domain. On a disk or on an annulus, the same behaviour can be observed, but these geometries require the inclusion of ghost vortices to maintain the boundary conditions. The boundedness of the domain also implies that solutions remain bounded for almost any initial condition when considering heterogeneous systems, i.e. systems with both positive and negative circulation vortices.

We express the barotropic quasi-geostrophic equations on the sphere [28] in terms of the stream function ψ and potential vorticity q

$$q_t + J(\psi, q) = 0 \tag{1}$$

$$q = \Delta_S \psi + 2\Omega z + h, \tag{2}$$

where Ω is the angular velocity of the sphere about the z-axis and h represents topography. The Laplace–Beltrami operator on the sphere Δ_S is defined (in spherical coordinates) as

$$\Delta_S \psi = \frac{1}{\cos\theta}\left[\frac{1}{\cos\theta}\psi_{\phi\phi} + \frac{\partial}{\partial\theta}\left(\cos\theta\,\psi_\theta\right)\right],$$

where ϕ is the longitude and θ the latitude. The Jacobian $J(f, g)$ is defined as

$$J(f, g) = \frac{1}{\cos\theta}\left(f_\phi g_\theta - g_\phi f_\theta\right). \tag{3}$$