



Solving Fredholm integral equations via a piecewise linear maximum entropy method



Congming Jin^a, Jiu Ding^{b,*}

^a Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China

^b Department of Mathematics, The University of Southern Mississippi, Hattiesburg, MS 39406-5045, USA

ARTICLE INFO

Article history:

Received 24 September 2014

Received in revised form 5 August 2015

MSC:

41A35

65D07

65J10

Keywords:

Fredholm integral equation

Maximum entropy

Piecewise linear approximations

ABSTRACT

We propose a piecewise linear approximation method, based on the maximum entropy principle, to approximate a nonnegative solution of a Fredholm integral equation numerically. The theoretical analysis and numerical examples show that our method has a convergence rate of order 2, and it can get more accurate approximations with more moments used without ill-condition of the classic maximum entropy approach. The method can also be applied to solve Fredholm integral equations with singular kernels.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

Recently a maximum entropy method using piecewise linear functions has been developed for recovering a density function for the statistical study of deterministic dynamical system on an interval [1]. This approach has its origin in the 1957 paper [2] by Jayne and was first adopted in [3] for the computation of a stationary density of an interval mapping $S : [0, 1] \rightarrow [0, 1]$. The renewed interest in the maximum entropy scheme is due to the observation that the employment of piecewise polynomial functions, which are polynomial functions on each sub-domain from a finite partition of the whole domain, can effectively eliminate the difficulty of ill-condition of the traditional maximum entropy method that uses only polynomials as moment functions. Because of the locally nonzero property of the basis for such functions, the finite element spaces of piecewise polynomial functions are widely used in numerical partial differential equations and integral equations.

Encouraged by the efficiency of the new finite element maximum entropy method, we would like to extend the approach to compute a nonnegative solution of a Fredholm integral equation numerically under the assumption that such a solution does exist.

A maximum entropy method was first proposed in [4] for solving Fredholm integral equations. Like all other traditional maximum entropy approaches for calculating a density function numerically, the moment related functions used in [4] were polynomials. Since the polynomial involved numerical computation is highly ill-conditioned as usual, it is not surprising that no more than 10 or so moments could be used in the computation of the examples in the paper. In the last section of [4] the author raised some question of the method: “It is not clear whether or not solutions generated by a larger number of moments than used here would improve the numerical accuracy significantly”. Our motivation of writing this paper is for answering the above question.

* Corresponding author.

E-mail address: jiudin@gmail.com (J. Ding).

After giving some preliminaries in the next section, we present our piecewise linear maximum entropy method for the integral equation in Section 3. A convergence analysis will be presented in Section 4. Numerical examples will be demonstrated in Section 5 and we conclude in Section 6.

2. Preliminaries

In this section we present the Fredholm integral equation and obtain an explicit formula for the solution of the maximum entropy problem.

Let a and b be two real numbers with $a < b$, let $L^1(a, b)$ be the Banach space of real-valued Lebesgue integrable functions defined on (a, b) with the L^1 -norm $\|f\|_1 = \int_a^b |f(x)|dx$, and let K be a linear integral operator from $L^1(a, b)$ into itself defined by

$$(Kf)(x) = \int_a^b k(x, y)f(y)dy,$$

where the kernel function k is measurable and bounded on its domain. Let $g \in L^1(a, b)$ be given. The following operator equation

$$(I - K)f = g \quad (1)$$

is called the *Fredholm integral equation of the second kind*. For the purpose of the applicability of the maximum entropy method for finding a solution of (1), we assume that a nonnegative solution f^* exists for the given g . A sufficient condition to make it happen is that g is a nonnegative function and K is a positive operator such that $\|K\|_1 < 1$, where $\|K\|_1$ is the operator norm of K . This follows from the Neumann series

$$(I - K)^{-1} = I + K + K^2 + \dots$$

so that $(I - K)^{-1}$ exists and is positive.

For any $f \in L^1(a, b)$ and $h \in L^\infty(a, b)$, the Banach space of essentially bounded Lebesgue measurable functions, we write $\langle f, h \rangle = \int_a^b f(x)h(x)dx$. Denote by $K' : L^\infty(a, b) \rightarrow L^\infty(a, b)$ the *dual operator* of K , defined by

$$\langle Kf, h \rangle = \langle f, K'h \rangle$$

for all $f \in L^1(a, b)$ and $h \in L^\infty(a, b)$. Then we have

$$(K'h)(y) = \int_a^b k(x, y)h(x)dx.$$

Let $h_1, \dots, h_s \in L^\infty(a, b)$ be a set of linearly independent functions. Multiplying each of them to the equality

$$(I - K)f^* = g$$

and integrating over $[a, b]$, we have

$$\langle f^*, (I - K')h_i \rangle = \langle g, h_i \rangle, \quad i = 1, \dots, s.$$

We call such initially chosen functions h_i the *generating functions*.

Define $g_i = (I - K')h_i$ for each i , which are called the *moment functions*. Denote $m_i = \langle g, h_i \rangle$ for each i . Then the above equalities can be written as the moment conditions

$$\langle f^*, g_i \rangle = m_i, \quad i = 1, \dots, s.$$

Under the condition that $f^* \in L^p(a, b)$ for some $1 < p < \infty$, if an infinite sequence of the functions $\{h_i\}$ is chosen such that the resulting sequence $\{g_i\}$ is dense in $L^q(a, b)$ with $1/p + 1/q = 1$, then the condition

$$\langle f^*, g_i \rangle = m_i, \quad i = 1, 2, \dots$$

determines f^* uniquely. This is possible, for example, if $I - K$ is one-to-one and onto, so that $(I - K)^{-1}$ is also bounded by Banach's Inverse Mapping Theorem, and if $\{h_i\}$ is dense in $L^q(a, b)$. The convergence analysis in Section 4 even shows that the sequence of piecewise linear functions satisfies the above purpose.

Our strategy to recover the unknown solution f^* numerically is via solving the equations

$$\langle w, g_i \rangle = m_i, \quad i = 1, \dots, s$$

for the unknown nonnegative function w with the maximum entropy principle. In the maximum entropy approximation, for the convenience of notation and analysis, we maximize a *modified Boltzmann entropy* [5], which is a nonlinear functional

Download English Version:

<https://daneshyari.com/en/article/4637974>

Download Persian Version:

<https://daneshyari.com/article/4637974>

[Daneshyari.com](https://daneshyari.com)