Journal of Computational and Applied Mathematics 300 (2016) 56–67

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Degree elevation of changeable degree spline

Wanqiang Shen *, Ping Yin, Chengjie Tan

School of Science, Jiangnan University, Wuxi City, Jiangsu Province, 214122, China

ARTICLE INFO

Article history:
Received 5 March 2014
Received in revised form 23 November 2015

Keywords:
Computer aided geometric design
B-spline
Basis function
Degree elevation
Changeable degree spline
Corner cutting

ABSTRACT

Changeable degree spline (CD-spline), a direct extension of the B-spline, allows different segments to possess different degrees. In this paper, we present a method for degree elevation of a CD-spline. The method is flexible, as it applies to any segment of any CD-spline elevated by any degree. A CD-spline can be transformed into a B-spline, and vice versa, in various orders by using this method. Moreover, this degree elevation method is presented as a corner cutting process.

1. Introduction

A changeable degree spline (CD-spline) is derived from the idea of piecewise curve made up of polynomial segments of variable degrees. Earlier, the variable degree polynomial curves [1–3], kinds of generalized B-splines [4], as each segment is determined by four control points, are like cubic B-splines. Later, Ref. [5] proposes the requirement of the compatibility with the B-spline for geometric modeling and each n-degree segment should be given by $n+1$ control points. It builds some piecewise polynomial curves of degrees 1, 2, 3, which are called multi-degree splines. In 2007, Ref. [6] introduces a bi-degree spline made up of segments of degrees n and $n+1$. It is a special case of the CD-spline, introduced in [7] in 2010, allowing any different degrees for any different segments.

The CD-spline, possesses many B-spline-like modeling properties, such as convex hull, local control, geometric invariance and continuous property. The knots can be inserted in its knot sequence and it can be evaluated recursively. Its basis functions, like the B-spline, build the unique normalized B-basis [8], which possesses the optimal shape preserving property in their spanned space. Therefore, the CD-spline is a direct extension of the B-spline. Compared with the B-spline, different degree segments may result in less data. For example, a line segment is connected with a cubic curve. If using the B-spline, then the line segment should be regarded as a cubic curve and needs four control points. If using the CD-spline, then only two control points are needed for the line segment. So, the CD-spline may reduce the redundant data for modeling curves. Moreover, it includes the B-spline as a subcase, which may be compatible with the current B-spline system in theory.

However, a CD-spline has two significant drawbacks. First, the recursive integral definition motivates [9] as an attempt of providing a representation in terms of determinants. It may be used to prestore the CD-spline basis functions and avoid computational problem in executive programs. Secondly, there are no recurrence relations for degree elevation of a curve defined by its basis functions. This paper will explore this further.

Degree elevation is an important capability for any curve modeling system. It can be used to approximate a curve, raise a curve's freedom and conjoin two curves. Regarding the conventional B-splines, earlier researches on degree elevation [10–15] show no geometric means of corner cutting. However, corner cutting algorithms are attractive, since they provide

* Corresponding author.
E-mail address: wq_shen@163.com (W. Shen).

http://dx.doi.org/10.1016/j.cam.2015.11.030

simple geometric constructions in computer aided geometric design [16]. Until 2007, the bi-degree spline [6] solves the corner cutting problem of the B-spline. Similar methods are used for degree elevation of a C-B-spline [17] and a UE-spline [18], which are both B-spline-like curves for the space combined with algebraic and trigonometric/hyperbolic polynomials.

Throughout that research, the degree elevation property of CD-spline appears only in the special case of bi-degree [6], and the degree is only increased by 1. Moreover, the elevation order in the degree elevation process of the B-spline is fixed, from left to right. In this paper, we consider more general cases. Our degree elevation method for the CD-spline will:

- apply to any segment of any CD-spline;
- apply to any increased degree;
- provide variable orders in the elevation process;
- present a (step-by-step) corner cutting process.

In the following section, we review the definition and the relevant properties of a CD-spline. In Section 3, we state the degree elevation method for an arbitrary segment of a CD-spline. First we consider the degree increased by 1, followed by considering an increase of any positive integer μ. We will provide the transforming algorithms, based on degree elevation, between the B-spline and the CD-spline in Section 4. Conclusions are drawn in the final section.

2. CD-spline [7]

Different segments of a CD-spline may have different degrees, so we need to know both the knot sequence and the corresponding degree sequence. Let $\boldsymbol{T} := \{t_i\}_{i\in\mathbb{Z}}$ be a nondecreasing real number sequence and $\boldsymbol{G} := \{d_i\}_{i\in\mathbb{Z}}$ be a bounded positive integer sequence satisfying the following condition:

(C) If $t_{i-1} < t_i = t_{i+1} = \cdots = t_{i+m-1} < t_{i+m}$, then $d_i = d_{i+1} = \cdots = d_{i+m-1}$ and $\max\{1, d_i - d_{i-1} + 1\} \le m \le d_i$.

Then, $\boldsymbol{T} = \{t_i\}_{i\in\mathbb{Z}}$ is called a knot sequence, and $\boldsymbol{G} = \{d_i\}_{i\in\mathbb{Z}}$ is called a degree sequence of $\boldsymbol{T} = \{t_i\}_{i\in\mathbb{Z}}$. That is, the knot interval $[t_i, t_{i+1})$ has the degree d_i. We call it a d_i-interval for simplicity. A knot of multiplicity m is called an m-knot. For example, if $t_{i-1} < t_i = t_{i+1} = \cdots = t_{i+m-1} < t_{i+m}$, then $t_i, t_{i+1}, \ldots, t_{i+m-1}$ are all m-knots.

Throughout the remainder of this paper, we always assume that $\boldsymbol{T}$ is the knot sequence $\{t_i\}_{i\in\mathbb{Z}}$ and $\boldsymbol{G}$ is the degree sequence $\{d_i\}_{i\in\mathbb{Z}}$ of $\boldsymbol{T}$ fulfilling the Condition **C**, and $D := \max_i \{d_i\}$.

For $n = 0, 1, \ldots, D$, functions $N_{i,n} := N_{i,n}(t)$, $i \in \mathbb{Z}$, over $\boldsymbol{T}$ and $\boldsymbol{G}$, are generated by the following iterative method. The resulting functions $N_{i,D}$, $i \in \mathbb{Z}$, are the CD-spline basis functions over $\boldsymbol{T}$ and $\boldsymbol{G}$. For simplicity, we set $n_i := d_i - D + n$ in the whole paper.

$$N_{i,n}(t) := \begin{cases} 0, & \text{if } n_i < 0, \\ \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}), \\ 0, & \text{otherwise}, \end{cases} & \text{if } n_i = 0, \\ \displaystyle\int_{-\infty}^{t} \left[\delta_{i,n-1} N_{i,n-1}(s) - \delta_{i+1,n-1} N_{i+1,n-1}(s)\right] ds, & \text{if } n_i > 0, \end{cases} \tag{1}$$

where $\delta_{i,n} := \left(\int_{-\infty}^{+\infty} N_{i,n}(t)\, dt\right)^{-1}$. Note that (1) includes the definition of the initial functions $N_{i,0}$, $i \in \mathbb{Z}$, since $n_i \le 0$ if $n = 0$. Moreover, $\delta_{i,n} N_{i,n}$ is the Dirac function if $N_{i,n} = 0$. That is,

$$\int_{-\infty}^{t} \delta_{i,n} N_{i,n}(s)\, ds = \begin{cases} 0, & \text{if } t < t_i, \\ 1, & \text{if } t \ge t_i. \end{cases}$$

Some of the B-spline-like properties of the CD-spline basis functions are as follows:

Property 2.1 (*Normalization*)**.** *For any $t \in \mathbb{R}$, the function*

$$\sum_{i=-\infty}^{+\infty} N_{i,D} = 1. \tag{2}$$

Property 2.2 (*Local Support*)**.** *$N_{i,D}$ is a positive function supported on $[t_i, t_{i+k})$, where $k := k_{i,D}$ and the sequence $\{k_{i,n}\}_{i\in\mathbb{Z}}$ is recursively defined by*

$$k_{i,n} := \begin{cases} 0, & \text{if } n_i < 0, \\ 1, & \text{if } n_i = 0, \\ \begin{cases} k_{i,n-1}, & \text{if } d_{i+1} < D - n + 1, \\ k_{i+1,n-1} + 1, & \text{if } d_{i+1} \ge D - n + 1, \end{cases} & \text{if } n_i > 0. \end{cases} \tag{3}$$