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A stable and scalable hybrid solver for rate-type non-Newtonian fluid models

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ABSTRACT

We present and analyze hybrid discretization schemes for rate-type non-Newtonian fluids models. The method employs higher order conforming approximations for velocity and pressure of the Stokes equation and lower order approximations such as piecewise linear or piecewise constant for conformation tensor. To reduce the accuracy gap, the constitutive equation is discretized on a refined mesh, which is obtained by subdividing the mesh for the velocity and pressure. The temporal discretization is made by the standard semi-Lagrangian scheme. The fully discrete nonlinear system is shown to be solved iteratively by applying the three steps:(1) locating the characteristic feet of fluid particles, (2) solving the constitutive equation, and (3) solving the momentum and continuity equations (Stokes-type equation). To achieve a scalability of the solution process, we employ an auxiliary space preconditioning method for the solution to the conforming finite element methods for the Stokes equation. This method is basically two-grid method, in which lower-order finite element spaces are employed as auxiliary spaces. It is shown to not only lead to the mesh independent convergence, but also improve robustness and scalability. Stability analysis shows that if $\Delta t = O(h^d)$, where d is the dimension of domain, then the scheme admits a globally unique solution. A number of full 3D test cases are provided to demonstrate the advantages of the proposed numerical techniques in relation to efficiency, robustness, and weak scalability.

1. Introduction

We consider a creeping fluid that occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$. Let $\mathbf{f} \in \left(L^2(\Omega)\right)^d$ be a given function. The dimensionless form of the momentum balance and continuity equations can be written as

$$-\mu_s \Delta \mathbf{u} + \nabla p = \nabla \cdot \boldsymbol{c} + \mathbf{f} \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.1b}$$

subject to certain boundary conditions for $\mathbf{u}$ and $\boldsymbol{c}$. Note that μ_s is the Newtonian viscosity, $\mathbf{u}$ is the velocity, p is the pressure and $\boldsymbol{c}$ is the extra stress tensor. The non-trivial extra stress $\boldsymbol{c}$, called the conformation tensor, makes Eq. (1.1) different from the classical Newtonian case (the Stokes equation). An often-used non-Newtonian constitutive model is the so-called

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Oldroyd-B model [1]:

$$\boldsymbol{c} + \mathrm{Wi}\frac{\delta_F \boldsymbol{c}}{\delta_F t} = \frac{\mu_p}{\mathrm{Wi}}\boldsymbol{\delta}, \tag{1.2}$$

where Wi is the Weissenberg number, which controls the fluids elasticity [2], μ_p is the polymeric viscosity, $\boldsymbol{\delta}$ is the identity tensor and $\delta_F \boldsymbol{c}/\delta_F t$ is the upper convected derivative of $\boldsymbol{c}$ [3], namely

$$\frac{\delta_F \boldsymbol{c}}{\delta_F t} := \frac{D\boldsymbol{c}}{Dt} - \nabla \mathbf{u}\, \boldsymbol{c} - \boldsymbol{c}\,(\nabla \mathbf{u})^T = \frac{\partial \boldsymbol{c}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{c} - \nabla \mathbf{u}\, \boldsymbol{c} - \boldsymbol{c}\,(\nabla \mathbf{u})^T, \tag{1.3}$$

where $D\boldsymbol{c}/Dt = \partial \boldsymbol{c}/\partial t + \mathbf{u} \cdot \nabla \boldsymbol{c}$ is called the material time derivative of $\boldsymbol{c}$. There are abundance of the rate-type models in literature, which include Johnson–Segalman models [4], Giesekus model [5] and Phan-Thien Tanner models [6]. In this paper, we shall propose numerical methods limited to the Oldroyd-B model. However, they can be equally well applied to other rate type models.

Many researchers have proposed many different discretization schemes for Eqs. (1.1) and (1.2). Among others, readers refer to [7,1,8] and references therein. The main purpose of this paper is to extend and improve the method originally introduced in [9,1]. The proposed method in [9,1] can be summarized as follows. We apply the semi-Lagrangian scheme for the material derivative $D\boldsymbol{c}/Dt$ along the characteristics [10,11], i.e.

$$\frac{D\boldsymbol{c}(x,t)}{Dt} \approx \frac{\boldsymbol{c}(x,t) - \boldsymbol{c}(x^*, t-\Delta t)}{\Delta t}, \tag{1.4}$$

where Δt is the time step size and x^* is the characteristic feet of x at time $t - \Delta t$. Note that the characteristic feet x^* can be obtained by solving the following ordinary differential equation for the flow map $y : \Omega \mapsto \Omega$, backward in time:

$$\frac{dy(x,s)}{ds} = \mathbf{u}(y(x,s), s) \quad \text{with } y(x,t) = x. \tag{1.5}$$

The fully-discrete system can then be obtained by introducing spatial discretizations. In particular, we employ finite element pairs denoted by $\mathbf{V}_\hbar$ for $\mathbf{u}$ and $W_\hbar$ for p such that $\nabla \cdot \mathbf{V}_\hbar \subseteq W_\hbar$ and inf–sup condition [12–14] holds true and employ piecewise constant approximation for the conformation tensor $\boldsymbol{c}$ to arrive at the following fully discrete nonlinear equation as follows:

$$-\mu_s \Delta_\hbar \mathbf{u}_\hbar + \nabla_\hbar p_\hbar = \nabla_\hbar \cdot \boldsymbol{c}_\hbar + \mathbf{f}_\hbar \tag{1.6}$$

$$\nabla_\hbar \cdot \mathbf{u}_\hbar = 0 \tag{1.7}$$

$$\left(\frac{\boldsymbol{c}_\hbar}{\Delta t} - \frac{\boldsymbol{c}_0}{\Delta t}\right) - \nabla_\hbar \mathbf{u}_\hbar\, \boldsymbol{c}_\hbar - \boldsymbol{c}_\hbar\,(\nabla_\hbar \mathbf{u}_\hbar)^T = \frac{\mu_p}{\mathrm{Wi}^2}\boldsymbol{\delta} - \frac{1}{\mathrm{Wi}}\boldsymbol{c}_\hbar, \tag{1.8}$$

where $\boldsymbol{c}_0 = \boldsymbol{c}_\hbar(x^*, t - \Delta t)$. It is shown that the fully discrete equations (1.6)–(1.8) are shown to be energy stable in [9,1] under the condition that the integral of the conformation tensor at the characteristic feet is computed accurately. Using the stability in energy norm, the global existence of the discrete solution has been established in [15] as well under the assumption that the time step size Δt is chosen so that $\Delta t = O(\hbar^d)$. Note that if conforming finite element pairs $\mathbf{V}_\hbar$ and $W_\hbar$ are chosen such that $\nabla \cdot \mathbf{V}_\hbar \subseteq W_\hbar$ and inf–sup stability holds true, then typically the degree of polynomials for $\mathbf{V}_\hbar$ should be of higher order (degree at least 4 in 2D [14] and degree at least 6 in 3D [16]). On the other hand, the stability for the full equation is established under the assumption that the conformation tensor is approximated by a piecewise constant function. Therefore, there is an apparent gap in the approximation accuracy between the velocity and the conformation tensor in this setting.

The main purpose of this paper is in eliminating such an accuracy gap inherent in the algorithms discussed in [9,15,1]. More precisely, given a mesh of size h for the velocity and pressure, we subdivide the target mesh of size h to obtain a refined mesh of size $\hbar$ on which the constitutive equation is discretized, thereby obtaining a hybrid discretization for the rate-type non-Newtonian equations. The lower order finite element for the conformation tensor is then defined on the refined mesh. We show that under the same assumptions given in our previous works [9,1,15], the proposed hybrid discrete system is energy-stable. Furthermore, our extension includes the result that the stability will be intact even if the piecewise linear element is employed for the conformation tensor and the global existence of the discrete solution can be achieved if $\Delta t = O(h^d)$ unlike the single grid case in which $\Delta t = O(\hbar^d)$ should be a sufficient condition. We remark that the use of linear elements for the conformation tensor $\boldsymbol{c}_h$ is already introduced and proven to be stable in relation to the free energy estimate, [17]. However, their formulation is different from our approach in the characteristic interpolation step and uses Petrov–Galerkin type weak formulation for the constitutive equations. Furthermore, their approximation scheme is not hybrid. The proposed hybrid discretization seems to be new for the solution to the non-Newtonian flow models to the authors' best knowledge.

In addition to such theoretical contributions of this paper, we present a weakly scalable solution scheme for the coupled nonlinear system (1.6)–(1.8) as well. First of all, the characteristic method is shown to be quite effective to solve the constitutive equation in the hybrid methods. Note that with the usage of this algorithm, since Eq. (1.8) is local in nature, a fully parallel algorithm can be devised for solution of $\boldsymbol{c}_\hbar$ with no significant difficulty. We present weak scalable methods to