Journal of Computational and Applied Mathematics 300 (2016) 119–133

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A solution method for autonomous first-order algebraic partial differential equations

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ARTICLE INFO

Article history:
Received 3 March 2015
Received in revised form 18 November 2015

MSC:
68W30
35F20
35A30
14J70

Keywords:
Partial differential equations
Algebraic hypersurface
Rational parametrization
Complete solution

ABSTRACT

In this paper we present a procedure for solving first-order autonomous algebraic partial differential equations in an arbitrary number of variables. The method uses rational parametrizations of algebraic (hyper)surfaces and generalizes a similar procedure for first-order autonomous ordinary differential equations. In particular we are interested in rational solutions and present certain classes of equations having rational solutions. However, the method can also be used for finding non-rational solutions.

1. Introduction

The problem of finding exact solutions to partial differential equations has been deeply studied in the literature. However, there is not a general method to be followed when handling a specific equation but different methods and techniques which might be applied to solving equations of certain form or under some assumptions on the elements involved. We provide [1, Sec. II.B] as a reference in this direction.

The main aim of the present work is to provide an alternative novel exact method for solving partial differential equations. More precisely, we study algebraic partial differential equations (APDEs) which are autonomous and of first-order (see Section 2 for a precise definition).

1.1. Algebro-geometric treatment of AODEs—state of the art

Recently algebraic–geometric solution methods for algebraic ordinary differential equations (AODEs) were investigated. First results on solving first order AODEs can be found in [2] where Gröbner bases are used and [3] where a degree bound is computed which might be used for making an ansatz. The starting point for algebraic–geometric methods, such as the one described in this paper, was an algorithm by Feng and Gao [4,5] which decides whether or not an autonomous AODE,

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http://dx.doi.org/10.1016/j.cam.2015.12.030

$F(y, y') = 0$ has a rational solution and in the affirmative case computes a rational general solution. This result was then generalized by Ngô and Winkler [6–8] to the non-autonomous case $F(x, y, y') = 0$. First results on higher order AODEs can be found in [9–11]. Ngô, Sendra and Winkler [12] also classified AODEs in terms of rational solvability by considering affine linear transformations. A generalization to birational transformations can be found in [13]. In [14,15] a solution method for autonomous AODEs is presented which generalizes the method of Feng and Gao to finding radical and also non-radical solutions. A generalization of the procedure to APDEs in two variables can be found in [16]. In this paper we present a further generalization to the case of an arbitrary number of variables.

1.2. The novel solution method proposed

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, and let $F \in \mathbb{K}(x_1, \dots, x_n)\{u\}$ be an element of the ring of differential polynomials in u (derivatives are w.r.t. $x_1, \dots, x_n$), which is also a polynomial in $x_1, \dots, x_n$. For our purposes we assume:

(a) F only depends on the first order derivatives of u w.r.t. $x_1, \dots, x_n$, say $u_{x_1}, \dots, u_{x_n}$.
(b) F does not depend on $x_1, \dots, x_n$.

Under the previous assumptions we may write the first-order autonomous APDE associated to F as

$$F(u, u_{x_1}, \dots, u_{x_n}) = 0. \quad (1)$$

Our method aims to solve such equations.

The method departs from a given proper rational parametrization of the hypersurface $F(z, p_1, \dots, p_n) = 0$, say

$$\mathcal{Q}(s_1, \dots, s_n) = (q_0(s_1, \dots, s_n), q_1(s_1, \dots, s_n), \dots, q_n(s_1, \dots, s_n)).$$

We assume that the parametrization can be expressed in the form

$$\mathcal{Q}(s_1, \dots, s_n) = \mathcal{L}(g(s_1, \dots, s_n)),$$

where g is an invertible map. If we are able to compute $h = g^{-1}$, then

$$\mathcal{Q}(g^{-1}(s_1, \dots, s_n)) = \mathcal{L}(s_1, \dots, s_n),$$

provides a solution $q_1(h_1, \dots, h_n)$ of (1).

The solution method, algorithmically described in Procedure 1, makes use of the method of characteristics applied to an auxiliary system of quasilinear equations (see (8)). If it returns a function, it is a solution of the initial problem (see Theorem 3.6).

It is worth remarking a distinguished difference with respect to the case of ordinary differential equations. Whilst a non-constant solution of an autonomous AODE always provides a proper parametrization of the associated curve, this is no longer valid when working with APDEs. The method provides, if it successfully arrives to a rational solution of the APDE under study, a proper solution of suitable dimension (see Definitions 2.2 and 2.3 for the definitions of proper solution and solution of suitable dimension respectively, and Theorem 4.2 for this statement).

Another important feature under discussion in the work is completeness of the solution (see Definition 2.2) which is also attained. From the knowledge of a complete solution one can construct any other solution of the problem (see [17]), so we only focus our results on those.

Our method provides a tool for systematically solving various well-known equations (see Table 2). Some of them are enumerated throughout the text, such as those studied in the examples in Section 4. Moreover, the algorithm may be applied to find other solutions rather than just rational ones (see Section 5).

1.3. Main contributions of the paper

The main contribution of the paper is the development of a novel method to find exact solutions for autonomous first-order algebraic partial differential equations. The value of this approach is motivated by the following facts:

- In our method, if it returns a rational function, then this function provides a solution to the autonomous first-order ADPE under study. Moreover, this solution turns out to be proper, complete, and of suitable dimension.
- Our method is not restricted to obtain rational solutions of APDEs (see Section 5).
- Our method generalizes known results obtained in the framework of ODEs. It gives solutions to some well-known partial differential equations studied in the literature.
- Even if the method fails, it often leads to an implicit description of the solution.