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A generalized eigenvalue algorithm for tridiagonal matrix pencils based on a nonautonomous discrete integrable system

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ABSTRACT

A generalized eigenvalue algorithm for a certain class of tridiagonal matrix pencils is presented. The algorithm appears as the time evolution equation of a nonautonomous discrete integrable system associated with a polynomial sequence which has some orthogonality on the support set of the zeros of the characteristic polynomial for a tridiagonal matrix pencil. The convergence of the algorithm is discussed by using the solution to the initial value problem for the corresponding discrete integrable system.

1. Introduction

Applications of discrete integrable systems to numerical algorithms are important and fascinating topics. Since the end of the twentieth century, a number of relationships between classical numerical algorithms and integrable systems have been studied (see the review papers [1–3]). On this basis, new algorithms based on discrete integrable systems have been developed: (i) singular value algorithms for bidiagonal matrices based on the discrete Lotka–Volterra equation [4,5], (ii) Padé approximation algorithms based on the discrete relativistic Toda lattice [6] and the discrete Schur flow [7], (iii) eigenvalue algorithms for band matrices based on the discrete hungry Lotka–Volterra equation [8] and the nonautonomous discrete hungry Toda lattice [9], and (iv) algorithms for computing D-optimal designs based on the nonautonomous discrete Toda (nd-Toda) lattice [10] and the discrete modified KdV equation [11].

In this paper, we focus on a nonautonomous discrete integrable system called the R_{II} *chain* [12], which is associated with the generalized eigenvalue problem for tridiagonal matrix pencils [13]. The relationship between the finite R_{II} chain and the generalized eigenvalue problem can be understood to be an analogue of the connection between the *finite nd-Toda lattice* and the eigenvalue problem for tridiagonal matrices. In numerical analysis, the time evolution equation of the finite nd-Toda lattice is called the *dqds (differential quotient difference with shifts) algorithm* [14], which is well known as a fast and accurate iterative algorithm for computing eigenvalues or singular values. Therefore, it is worth to consider the application of the finite R_{II} chain to algorithms for computing generalized eigenvalues. The purpose of this paper is to construct a generalized

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eigenvalue algorithm based on the finite R_{II} chain and to prove the convergence of the algorithm. Further improvements and comparisons with traditional methods will be studied in subsequent papers.

The nd-Toda lattice on a semi-infinite lattice or a non-periodic finite lattice has a *Hankel determinant solution*. In the background, there are *monic orthogonal polynomials*, which give rise to this solution; monic orthogonal polynomials have a determinant expression that relates to the Hankel determinant, and spectral transformations for monic orthogonal polynomials give the Lax pair of the nd-Toda lattice [15,16]. Especially, for the finite lattice case, we can easily solve the initial value problem for the nd-Toda lattice with the Gauss quadrature formula for *monic finite orthogonal polynomials*. This special property of the discrete integrable system allows us to analyse the behaviour of the system in detail and tells us how parameters should be chosen to accelerate the convergence of the dqds algorithm. We will give a review of this theory in Section 2.

The theory above will be extended to the R_{II} chain in Section 3. The three-term recurrence relation that monic orthogonal polynomials satisfy arises from a tridiagonal matrix. In a similar way, a tridiagonal matrix pencil defines a monic polynomial sequence. This polynomial sequence, called monic R_{II} *polynomials* [17], possesses similar properties to monic orthogonal polynomials and their spectral transformations yield the monic type R_{II} chain. A determinant expression of the monic R_{II} polynomials gives a Hankel determinant solution and, in particular for the finite lattice case, a convergence theorem of the monic R_{II} chain is shown under an assumption. This theorem enables us to design a generalized eigenvalue algorithm.

The dqds algorithm is a subtraction-free algorithm, i.e., the recurrence equations of the dqds algorithm do not contain subtraction operations except origin shifts (see Section 2.3). The subtraction-free form is numerically effective to avoid the loss of significant digits. In addition, there is another application of the subtraction-free form: ultradiscretization [18] or tropicalization [19]; e.g., the ultradiscretization of the finite nd-Toda lattice in a subtraction-free form gives a time evolution equation of the box–ball system with a carrier [20]. In Section 4, for the monic type R_{II} chain, we will present its subtraction-free form, which contains no subtractions except origin shifts under some conditions. It is considered that this form makes the computation of the proposed algorithm more accurate. At the end of the paper, numerical examples will be presented to confirm that the proposed algorithm computes the generalized eigenvalues of given tridiagonal matrix pencils fast and accurately.

2. Monic orthogonal polynomials, nd-Toda lattice, and dqds algorithm

First, we will review the connection between the theory of orthogonal polynomials and the nd-Toda lattice.

2.1. Infinite dimensional case

Let us consider a tridiagonal semi-infinite matrix of the form

$$B^{(t)} = \begin{pmatrix} u_0^{(t)} & 1 & & & \\ w_1^{(t)} & u_1^{(t)} & 1 & & \\ & w_2^{(t)} & u_2^{(t)} & 1 & \\ & & w_3^{(t)} & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \quad u_n^{(t)} \in \mathbb{C},\ w_n^{(t)} \in \mathbb{C} - \{0\},$$

where $t \in \mathbb{N}$ is the discrete time, whose evolution will be introduced later. Let I_n denote the identity matrix of order n and $B_n^{(t)}$ the nth order leading principal submatrix of $B^{(t)}$. We now introduce a polynomial sequence $\{\phi_n^{(t)}(x)\}_{n=0}^{\infty}$:

$$\phi_0^{(t)}(x) := 1, \qquad \phi_n^{(t)}(x) := \det(xI_n - B_n^{(t)}), \quad n = 1, 2, 3, \ldots.$$

By definition, $\phi_n^{(t)}(x)$ is a monic polynomial of degree n. The Laplace expansion for $\det(xI_{n+1} - B_{n+1}^{(t)})$ with respect to the last row yields the three-term recurrence relation

$$\phi_{n+1}^{(t)}(x) = (x - u_n^{(t)})\phi_n^{(t)}(x) - w_n^{(t)}\phi_{n-1}^{(t)}(x), \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where we set $w_0^{(t)} := 0$ and $\phi_{-1}^{(t)}(x) := 0$. It is well known that the three-term recurrence relation of the form (2.1) gives the following classical theorem.

Theorem 2.1 (*Favard's Theorem [21, Chapter I, Section 4]*). *For the polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^{\infty}$ satisfying the three-term recurrence relation (2.1) and any nonzero constant $h_0^{(t)}$, there exists a unique linear functional $\mathcal{L}^{(t)}$ defined on the space of all polynomials such that the orthogonality relation*

$$\mathcal{L}^{(t)}[x^m \phi_n^{(t)}(x)] = h_n^{(t)} \delta_{m,n}, \quad n = 0, 1, 2, \ldots, \ m = 0, 1, \ldots, n, \tag{2.2}$$

holds, where

$$h_n^{(t)} = h_0^{(t)} w_1^{(t)} w_2^{(t)} \cdots w_n^{(t)}, \quad n = 1, 2, 3, \ldots,$$

and $\delta_{m,n}$ is Kronecker delta.