Journal of Computational and Applied Mathematics 300 (2016) 172–182

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

An inverse eigenvalue problem for the finite element model of a vibrating rod

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ARTICLE INFO

Article history:
Received 3 December 2013
Received in revised form 10 November 2015

MSC:
15A18
65F15
65F18

Keywords:
Eigenvalue
Inverse problem
Rod
Finite element model

ABSTRACT

An inverse eigenvalue problem for the finite element model of a longitudinally vibrating rod whose one end is fixed and the other end is supported on a spring is considered. It is known that the mass and stiffness matrices are both tridiagonal for the finite element model of the rod based on linear shape functions. It is shown that the cross section areas can be determined from the spectrum of the rod. The inverse vibration problem can be recast into an inverse eigenvalue problem of a special Jacobi matrix. The necessary and sufficient conditions for the construction of a physically realizable rod with positive cross section areas are established. A numerical method is presented and an illustrative example is given.

1. Introduction

The process of formulating, analyzing and deriving the spectral information and, hence, inducing the dynamic behavior of a system from its priori known physical parameters, is referred to as a direct vibration problem. The inverse vibration problem, in contrast, is to validate, determine, or estimate the physical parameters of the system according to its observed or expected dynamical behavior. The inverse vibration problem is just as important as the direct vibration problem in applications. There are different kinds of inverse vibration problems depending on the type of system, the model of system and prescribed spectral data. Inverse vibration problems are studied extensively, see, for example, [1]. The discrete inverse problems in vibration may be transformed into inverse eigenvalue problems for structured matrices. See [2–4] for an overall treatment of inverse eigenvalue problems, [5–10] for an exhaustive classification of, and computational procedures for the inverse eigenvalue problems.

A system has a structure, a connectivity pattern, and this will be mirrored in the structures, the patterns of zero and non-zero entries in the corresponding matrix. For the problem to be more significant, it is often necessary to confine the reconstruction to certain special classes of matrices. On the other hand, lots of specially structured matrices enjoy many interesting properties. These properties not only can make the algorithms for some direct problems more efficient, for example, [11,12], but also can help to solve inverse eigenvalue problems. The inverse eigenvalue problem for the finite element model of a vibrating rod considered in this paper is a special case of the inverse eigenvalue problems of Jacobi matrices.

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http://dx.doi.org/10.1016/j.cam.2015.12.038

The free longitudinal vibrations of a thin rod of length L, cross-sectional area $A(x)$, Young's modulus E and density ρ are governed by the following equation

$$\frac{d}{dx}\left[EA(x)\frac{du(x)}{dx}\right] + \lambda\rho A(x)u(x) = 0, \quad 0 \le x \le L,\ \lambda = \omega^2, \tag{1}$$

where ω is the natural frequency of the rod. Assume that the left end of the rod is fixed and the right end is supported on a spring having the stiffness k_{n+1}. In this case, the end conditions are

$$u(0) = 0, \quad EA(L)\frac{du(x)}{dx}\bigg|_{x=L} = k_{n+1}u(L), \tag{2}$$

and are said to be the fixed–elastically restrained end conditions. The fixed–free and the fixed–fixed end conditions correspond to $k_{n+1} = 0$ and $k_{n+1} = \infty$, respectively.

Inverse eigenvalue problems of the continuous model of the rod are associated with inverse Sturm–Liouville problems, which have been addressed by numerous authors, see, for example, [13–18]. Levitan [19] and Gladwell [1] expounded the results associated with the inverse Sturm–Liouville problems. Ram [20] considered an inverse mode problem for the continuous model of an axially vibrating rod, and showed that the density and axial rigidity functions are determined by two eigenvalues, their corresponding eigenfunctions and the total mass of the rod. Recently, Gao [21], Morassi [22] presented numerical methods for recovering approximately the cross-sectional area of the vibrating rods having prescribed values of the first n eigenvalues.

Inverse vibration problems for the finite difference model of the vibrating rod are formulated as inverse eigenvalue problems for Jacobi matrices, which have been studied extensively, see, for example, [23–27]. Gladwell and Gbadeyan [28] showed that the mass and stiffness matrices of the finite difference model of the vibrating rod may be reconstructed uniquely from two sets of eigenvalues for the fixed–free and fixed–fixed boundary conditions. Gladwell [29] considered the inverse mode problem for the finite difference model of the vibrating rod, and showed that the discrete system may be constructed uniquely, apart from a scale factor, from two eigenvalues and corresponding eigenvectors. However, frequently, the physical properties, Young's modulus E and density ρ, of a homogeneous rod are constants and the cross-sectional area $A(x)$ varies. Ram and Elishakoff [30] showed that the cross-sectional area of the finite difference model of an axially vibrating non-uniform rod can be reconstructed from one eigenpair and the total mass of the rod. Ram and Elhay [31] considered the problem of reconstructing the cross-sectional area of the finite difference model of an axially vibrating rod from one set of eigenvalues for the fixed–free boundary condition, and proposed an iterative algorithm for solving the problem. Lu et al. [32, 33] recasted the problem into an inverse eigenvalue problem of a specially structured Jacobi matrix, and gave sufficient and some necessary conditions for the inverse eigenvalue problem to have a solution, and developed a numerical method for this problem.

Discretizing (1) and (2) by the finite element method, based on linear shape functions, we get

$$K_n u = \lambda M_n u, \tag{3}$$

where

$$K_n = \frac{nE}{L}\begin{pmatrix} A_1 + A_2 & -A_2 & & & \\ -A_2 & A_2 + A_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & A_{n-1} + A_n & -A_n \\ & & & -A_n & A_n + 3A_{n+1} \end{pmatrix},$$

and

$$M_n = \frac{\rho L}{6n}\begin{pmatrix} 2(A_1 + A_2) & A_2 & & & \\ A_2 & 2(A_2 + A_3) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2(A_{n-1} + A_n) & A_n \\ & & & A_n & 2A_n \end{pmatrix},$$

with

$$A_{n+1} = \frac{k_{n+1}L}{3nE}. \tag{4}$$

Note that the finite element model, based on linear shape functions, of Eq. (1) subject to the fixed–fixed end condition is given by

$$(K_{n-1} - \lambda M_{n-1})u = 0,$$