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Application of the generalized multiscale finite element method in parameter-dependent PDE simulations with a variable-separation technique

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ABSTRACT

In this paper, we combine the generalized multiscale finite element method (GMsFEM) with a variable-separation technique to tackle the parameter-dependent partial differential equations (PDEs). The solution is approximated via an expansion series, each term of which lives in the tensor product of the parametric space and the spatial space. Governing equations for each term are derived based on energy minimization. An iterative algorithm is presented to obtain the expansion series, which requires solving parameter-independent PDEs repeatedly. We then present the procedure of GMsFEM and apply it to these parameter-independent PDEs. Numerical examples are presented to demonstrate the effectiveness of the expansion series and the computational efficiency brought by GMsFEM.

1. Introduction

In many simulation scenarios in scientific research investigations and industrial production procedures, the governing partial differential equation (PDE) of the same physical process often needs to be solved simultaneously or repeatedly for various parameters. To give a few examples, in the problem of parameter estimation with PDE-constrained optimization [1], the same PDE needs to be solved for each update of the parameter; to solve stochastic PDEs with Monte Carlo methods [2,3], the same PDE needs to be solved for many samplings of the stochastic parameter variables; in reservoir management, porous media fluid flow simulation needs to be performed repeatedly with the time-evolving parameters under surveillance [4].

In cases as listed above, it can be advantageous to express the solution in a parameter-dependent form. Once we obtain this expression, the solution for another PDE simulation with a different parameter realization can be readily evaluated. Let us consider the prototype problem as presented in (1),

$$\begin{aligned} -\nabla \cdot (\kappa(x,\mu)\nabla u(x,\mu)) &= f(x) \quad \text{in } D;\\ u(x,\mu) &= 0 \quad \text{on } \partial D, \end{aligned} \tag{1}$$

where D denotes the spatial simulation domain with piecewise smooth boundary ∂D; $x \in D$ denotes the spatial variable; μ denotes the parameter variable, which is independent with x; κ denotes the coefficient and finally, u denotes the solution.

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We seek the solution of (1) in the mean sense, i.e., the solution u that satisfies

$$u = \underset{v}{\operatorname{argmin}} \int_{\mu} \int_{D} \left(\frac{1}{2} \kappa |\nabla v|^2 - f v \right) dx d\mu. \tag{2}$$

In order to solve for u numerically, we approximate u in the form of expression (3),

$$u \approx \sum_{i=1}^{N_{term}} a_i(\mu) v_i(x), \tag{3}$$

where $a_i(\mu)$ depends only on μ while $v_i(x)$ depends only on x. We emphasize here that for each term of the summation in (3), the parameter variable and the spatial variable are separated.

We will introduce an algorithm in Section 2 to solve for $a_i(\mu)$ and $v_i(x)$ of each term successively. A similar algorithm has been proposed in [5] to separate the solution's dependence on different spatial variables. Other related variable-separation techniques can be found in [6–11], among others. Intrinsically, we derive governing equations for each $a_i(\mu)$ and $v_i(x)$ by minimizing the energy associated with the original parameter-dependent PDE (1). By doing so, the original high dimensional problem is decomposed into sequences of low dimensional problems.

This strategy requires repeatedly solving PDEs of the same type as (1) with various μ-independent coefficients, which can still be prohibitively expensive. In the mean time, as will become clear in Section 2, the solution of each such PDE only constitute an iterate of $v_i(x)$, which in turn only contributes to one summation term in (3). Therefore, there is no need to solve these PDEs to an extreme level of accuracy. Instead, we apply the Generalized Multiscale Finite Element Method (GMsFEM) to these μ-independent PDEs to reduce the overall computational cost.

GMsFEM was first introduced in [12], which is closely related to the model reduction techniques presented in [13–15]. The main idea behind GMsFEM is to systematically enrich the initial coarse solution space using the eigenvectors of local spectral problems, which gradually accounts for more fine scale details. Since then, the methodology of GMsFEM has been augmented [16–21] and its application has been popularized [22–27].

The rest of this paper is organized as follows. In Section 2, we introduce the algorithm to solve for each term of u in expression (3) via iterations of parameter-independent PDE solving. In Section 3, we outline the application procedure of GMsFEM on solving these parameter-independent PDEs. In Section 4, numerical examples are presented to demonstrate the usefulness of the proposed algorithms. Finally, we conclude this paper with Section 5.

2. An iterative procedure for parameter-dependent PDEs

In this section, we present an algorithm to numerically solve PDE (1) in the mean sense (2), delivering the approximation in the form of (3). We note here that this algorithm is closely related to [5] and can be considered as belonging to the family of the Proper Generalized Decomposition (PGD) techniques [10,11].

To start, we derive the governing equations for the components of the first term in (3), i.e., $a_1(\mu)$ and $v_1(x)$, by asking the first term $a_1 v_1$ to minimize the objective function of (2), i.e.,

$$\{a_1, v_1\} = \underset{\{\tilde{a}_1, \tilde{v}_1\}}{\operatorname{argmin}} \int_{\mu} \int_{D} \left(\frac{1}{2} \kappa |\nabla \tilde{a}_1 \tilde{v}_1|^2 - f \tilde{a}_1 \tilde{v}_1 \right) dx d\mu. \tag{4}$$

By differentiating the objective function in (4) with respect to v_1 and a_1, respectively, we arrive at the following two coupled equations for v_1 and a_1:

$$-\nabla \cdot \left(\int_{\mu} \kappa a_1^2 d\mu \nabla v_1 \right) = \int_{\mu} f a_1 d\mu, \tag{5}$$

and

$$a_1 = \frac{\int_D f v_1 dx}{\int_D \kappa |\nabla v_1|^2 dx}. \tag{6}$$

Starting from some initial guess of a_1, we iteratively update the approximations of v_1 and a_1 by solving Eqs. (5) and (6) alternatively. When the change of a_1 and v_1 between two neighboring steps is small enough, we abort the iteration procedure.

After obtaining the approximations of a_1 and v_1, we substitute them into the expression (3), which is in turn substituted into the original PDE (1), arriving at the governing equation for the remaining of the solution $u - a_1 v_1 = \sum_{i=2}^{N_{term}} a_i v_i$. Obviously, the only difference between this new governing equation and the original PDE occurs at the right hand side, which is now $f + \nabla \cdot (\kappa \nabla a_1 v_1)$. Following the same strategy as we used for a_1 and v_1, we can now obtain the approximations for a_2 and v_2, i.e., the components of the second term in expression (3). We repeat the above procedure until the right hand side is no longer significant and arrive at the approximation in the form of (3). The above procedure is summarized in Algorithm 1.