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Multiscale approximation of the solution of weakly singular second kind Fredholm integral equation in Legendre multiwavelet basis

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ABSTRACT

Numerical solution of Fredholm integral equation of second kind with weakly singular kernel is obtained in this paper by employing Legendre multi-wavelet basis. The low- and high-pass filters for two-scale relations involving Legendre multiwavelets having four or five vanishing moments of their wavelets have been derived and are used in the evaluation of integrals for the multiscale representation of the integral operator. Explicit expressions for the elements of the matrix associated with the multiscale representation are given. An estimate for the Hölder exponent of the solution of the integral equation at any point in its domain is obtained. A number of examples is provided to illustrate the efficiency of the method developed here.

1. Introduction

Integral equations arise in various problems of mathematical physics in a natural way and as such they play important role in varieties of field of science and engineering [1–3]. Fredholm integral equation of the second kind with regular or singular kernels constitute an important branch in such area. Investigations on mathematical techniques for getting exact solution of such equations have been continued for a long time and reported in a number of literature [4–7]. However, such results are not adequate for getting solutions of integral equations appearing in several types of physical problems. Thus, investigations are continued simultaneously to find efficient approximation schemes which can provide approximate solution as accurate as possible. There already exist several numerical schemes for getting approximate solution of integral equations with singular kernels [8–19]. But in most of the cases the unknown function involved in the equation is expanded in some orthogonal bases so as to reduce the integral equation to a system of linear simultaneous equations for the unknown coefficients appearing in the truncated expansion of the function. However the matrices involved in the system of linear equations are dense and their condition numbers are high, in general. As a result their computational costs are prohibitive. Also the approximate solution when expanded in the basis consisting of classical orthogonal functions, often fails to provide

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local informations such as smoothness or regularity of the solutions, the error in the approximation in a straight forward manner. It is thus desirable to search for an appropriate technique which can provide the information on the aforesaid local behaviour or error in the approximation directly from the solution of the transferred algebraic equations at the expense of the less computational cost. It is now known that the use of wavelet basis of some multiresolution analysis(MRA) of underlying space of functions in approximating solution of integral equations can provide such informations efficiently [20–27].

Multiresolution analysis (MRA) of function spaces in terms of refinable functions and wavelets with compact support has significant applications in many areas of mathematical physics and engineering [28–32]. This can be attributed due to its elegant role of mathematical microscopy in the analysis of smoothness or regularity of a function. The main advantages of a numerical method involving expansion of the unknown function in terms of wavelets with compact support compared to a classical method are that discretization of the domain involved is inbuilt in the technique due to the compact support of the members in the basis and the resulting matrix is sparse due to vanishing moments of the wavelets. Consequently the numerical scheme involving basis of MRA becomes highly stable and the related computation is economic.

Alpert [20], Alpert et al. [21] are pioneers in the development of generation of wavelets involving polynomials and they used these to obtain numerical solutions of second kind integral equations with emphasis on logarithmically singular kernels. In recent years, several numerical methods for obtaining numerical solution of a Fredholm integral equation of the form

$$u(x)+\lambda\int_0^1\frac{u(t)}{|x-t|^{\mu}}dt=f(x),\quad 0<\mu<1,\ x\in[0,1] \tag{1.1}$$

where $f:[0,1]\to\mathbb{R}$ is the input function(known) and u is the output function(unknown), have been developed by a number of researchers [20,21,33,10,34–37]. Cai [37] observed that all existing numerical methods for solving weakly singular integral equations are of two types. In the first type, the integral equation is converted into a new but somewhat complicated integral equation possessing better regularity so as to be treated by some appropriate existing method [34]. The second type is called a hybrid projection method which allows the projection subspaces to contain some known singular functions [22,10]. This method requires a special attention for the evaluation of integrals involving singular functions in the basis. Since the basis (with compact support) of MRA of function spaces have inherent structure of regularization of singular functions, it is quite natural to investigate whether numerical solution of singular integral equation in the wavelet basis can achieve some additional benefits compared to existing classical methods.

In this paper Legendre multiwavelets have been used to solve a second kind Fredholm integral equation with weakly singular kernel. Definition of Legendre multiwavelets, their two-scale relations and associated filter coefficients are given in Section 2. Section 3 involves recurrence relations for integrals involving product of weakly singular kernels, scale functions and wavelets, some representative values of these integrals are also presented. Section 4 presents multiscale approximation of a function and multiscale representation of the weakly singular integral operator. In Section 5, the second kind integral equation is reduced to a linear system of algebraic equations involving multiscale representation of the integral operator and an estimate for Hölder exponent of the solution at any point within the domain is obtained. An error estimate for the approximate solution is obtained in Section 6. The numerical method developed here is illustrated through a number of examples in Section 7. Finally in Section 8 we present some concluding remarks including possible extension of the scheme to some other mathematical problems.

2. Legendre multiwavelets

The scaling functions in Legendre multiwavelet bases consist of K component vectors

$$\phi^i(x):=(2i+1)^{\frac{1}{2}}P_i(2x-1),\quad i=0,1,\ldots,K-1;\ 0\le x\le 1 \tag{2.1}$$

where $P_i(x)$ is the Legendre polynomial of degree i $(i=0,1,\ldots,K-1)$. At resolution j these are expressed as

$$\phi^i_{j,k}(x):=2^{\frac{j}{2}}\phi^i(2^jx-k),\quad j\in\mathbb{N} \tag{2.2}$$

where, $\mathbb{N}=\{0,1,2,\ldots\}$ and $supp\,\phi^i_{j,k}(x)=\left[\frac{k}{2^j},\frac{k+1}{2^j}\right)$. For a given $j>0$, shifting or translation of $\phi^i_{j,k}(x)$ is represented by the symbol k $\left(k=0,1,\ldots,2^j-1\right)$. In a particular resolution j, $\phi^{i_1}_{j,k_1}(x)$ is orthogonal to $\phi^{i_2}_{j,k_2}(x)$ for $i_1\ne i_2$, $k_1\ne k_2$ with respect to the inner product $\langle f,g\rangle=\int_0^1 f(x)\overline{g(x)}dx$. Apart from the usual recurrence relation (in n) for $P_n(x)$(viz. $(n+1)\,P_{n+1}(x)-(2n+1)\,x\,P_n(x)+n\,P_{n-1}(x)=0$), the refinement equations or the two-scale relations among the scale functions $\phi^i_{j,k}(x)$ are

$$\phi^i_{j,k}(x)=\frac{1}{\sqrt{2}}\sum_{r=0}^{K-1}\left(h^{(0)}_{i,r}\phi^r_{j+1,2k}(x)+h^{(1)}_{i,r}\phi^r_{j+1,2k+1}(x)\right)=\frac{1}{\sqrt{2}}\sum_{r=0}^{K-1}\sum_{s=0}^{1}h^{(s)}_{i,r}\phi^r_{j+1,2k+s}(x). \tag{2.3}$$