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# Simple-iteration method with alternating step size for solving operator equations in Hilbert space

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ABSTRACT

We introduce a new explicit iterative method with alternating step size for solving ill-posed operator equations of the first kind: $Ax = y$. We investigate the basic properties of the method for a positive bounded self-conjugate operator $A : H \to H$ in Hilbert space $H$ under the assumption that the error for the right part of the equation is available. We discuss the convergence of the method, for a given number of iterations, in the original Hilbert space norm, estimate its precision and formulate recommendations for choosing the stopping criterion. Furthermore, we prove the convergence of the method with respect to the stopping criterion and estimate the remaining error. In case the equation has multiple solutions, we prove that the method converges to the minimum norm solution.


## 1. Introduction

Linear operator equations $Ax = y$ abound in theoretical studies of machine learning, statistical signal processing, and applied mathematics [1]. Applications that give rise to linear operator equations include linear regression, optimal source allocation, optimal filtering, optimal control, and solutions to integral and partial differential equations, to name a few. Linear operator equations have been included into Gaussian Process (GP) regression and used for encoding physical and other background information into the measurement model [2]. The GP framework also led to methods for solving stochastic linear operator equations arising from noisy evaluations of the target function [3]. Linear operator equations have also been used for addressing ill-posed problems: when large deviations in solution $x$ correspond to small deviations in $y$. Such problems were originally signaled by Hadamard at the beginning of the previous century [4,5] but did not lead to further theoretical developments for a long time. Only when methods for solving ill-posed problems were introduced, and the correspondence to real world problems was made, the concept became adopted in countless engineering and physics applications. We refer to the method of quasi-solutions of Ivanov [6], the method based on the stopping criterion of Philips [7], and of course the widely used regularization method of Tikhonov [8]. To address the ill-posed problem of density estimation, support vector machines were used for solving linear operator equations by regularizing the $\varepsilon$-insensitive loss function, a method that uses non-Mercer's kernels [9]. This in turn led to application such as Bayes classifier implementations of multispectral data [10] and closed-form multiscale orthogonal projection operator wavelet kernels for analyzing very short-lived high frequency phenomena, such as transients in signals [11].

We focus on iterative methods for solving such equations when $A : H \to H$ is a positive bounded self-conjugate operator in Hilbert space $H$. Usually, the simple-iteration method is adopted with constant [12] or variable [13] step size which is

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required to depend on the sum of iterations. In order to reduce the required number of iterations, the step size needs to be as large as possible. However, the size of the steps has some restrictions [12,13]. In this paper, in order to mitigate these restrictions, we propose a new regularization method for solving ill-posed problems formulated as linear operator equations of the first kind. The method performs explicit iterations using alternating step sizes $\alpha$ and $\beta$ where $\beta$ does not have to satisfy the requirement of the variable step size method [13]. This renders our method much faster than comparable ones. The method relies on the most commonly adopted concept for solving of ill-posed problems: the Tikhonov regularizer. The proposed method is investigated in detail and compared with the literature.

We start developing the method given the a priori choice of the number of iterations. For the case where the right-hand side of the equation is available, sufficient conditions for convergence will be developed. Prior precision estimates will be obtained for a given class of problems $R(A)$ for which the solution $x$ is known given $y$. Albeit such information is usually unavailable or inaccurate, the prior choice of the number of iterations mostly has only theoretical value as it supports also in our case the identification of the fundamental properties of the proposed method.

In order to render the method more effective, even when no further information is available about the smoothness of the exact solution, we will further suggest a stopping criterion, and formulate recommendations how to select the number of iterations $n$ for which the residual error on $x$ is comparable to the accuracy of $y$. The synchronization of $n$ with respect to the accuracy of the right-hand side of the equation is usually called the small discrepancy principle. We will show that this leads to a method that is optimal for the said class of ill-posed problems. Furthermore, the choice of $n$ does not require any other information then the accuracy level of the right side of the equation. We will show that the proposed iterative method converges to the exact solution. Importantly, the estimate of the precision is obtained together with the stopping criterion.

Furthermore, the case of a non-unique solution will be investigated (i.e., for which the operator has a zero eigenvalue). We will prove that in this case the process converges to the solution with the smallest Hilbert space norm.

Finally, for a concrete case, we will show and compare the performance of our method with that of the simple-iteration method with variable step size before we draw our conclusions.

## 2. Iterative methods for solving linear operator equations

There are several reports on the fundamental properties of the simple iteration method $x_{n+1,\delta} = x_{n,\delta} + \alpha\left(y_\delta - Ax_{n,\delta}\right)$, $x_{0,\delta} = 0$ for solving linear operator equations of the first kind $Ax = y_\delta$ with $A : H \to H$ a positive bounded self-conjugate operator with $\|y - y_\delta\| \le \delta$. Konstantinova and Liscovec [12] showed that the method of simple iteration converges to the exact solution of the equation $Ax = y_\delta$ under condition that $0 < \alpha < \frac{2}{\|A\|}$ and with the number of iterations $n = n(\delta)$ depending on $\delta$ in such a way that $n(\delta)$, $\delta \to 0$ for $n \to \infty$, $\delta \to 0$. Given that $0 < \alpha \le \frac{5}{4\|A\|}$ and assuming that the exact solution can be written as $x = A^s z$, $s > 0$, the error can be estimated as $\left\|x - x_{n,\delta}\right\| \le s^s (n\alpha e)^{-s}\|z\| + n\alpha\delta$ which holds for all $n \ge 1$. It is clear that this estimate can be optimized for $n$. In addition, it can serve as a basis for the prior selection of the stopping moment. Also, for a given iterative procedure, the error was estimated and investigated for the case of an approximate operator $A_h : \|A_h\| \le h$. Taking into account the uncertainty of the operator, an estimate for the error was obtained: $\left\|x - y_{n,\delta}\right\| \le s^s (n\alpha e)^{-s}\|z\| + n\alpha\delta + ((1+\alpha h)^n - n\alpha h - 1)h^{-1}\|y_\delta\|$.

Bialy [14] proved convergence of his method for the case of a non-unique solution (0 is the eigenvalue of the operator). He showed that in this case the method converges to a solution with minimum norm. Emelin and Krasnoselskij [15] showed, for the first time, that the method of simple iteration $x_{n+1,\delta} = x_{n,\delta} + \alpha\left(y_\delta - Ax_{n,\delta}\right)$, $x_{0,\delta} = 0$ and the stopping rule based on discrepancy can be applied for solving the ill-posed equation of the first kind: $Ax = y_\delta$. The simple iteration principle was taken up by Vaynikko and Veretennikov [16] but with a different stopping rule based on discrepancy. They also investigated the error estimate and the estimate of the stopping moment in the case of an inaccurate right-hand side of the equation and an approximate operator.

Konstantinova and Liscovec [13] described a gradient-based method for finding the solution of the ill-posed equation $Ax = y_\delta$ using an alternating step size: $x_{n+1,\delta} = x_{n,\delta} - \alpha_{n+1}(Ax_{n,\delta} - y_\delta)$, $x_{0,\delta} = 0$. Under condition that $0 < \alpha_i < \frac{2}{\|A\|}$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, they proved convergence for the case of an approximate right-hand side, and derived a general estimate of the error in the case of an approximate operator.

In order to solve the ill-posed equation $Ax = y_\delta$, Matysik [17,21] proposed the following implicit and explicit iteration schemes:

$$x_{n+1,\delta} = \left(E - \alpha A^2\right)x_{n,\delta} + \alpha A y_\delta, \qquad x_{0,\delta} = 0; \qquad x_{n+1,\delta} = (E - \alpha A)^2 x_{n,\delta} + 2\alpha y_\delta - \alpha^2 A y_\delta, \qquad x_{0,\delta} = 0;$$

$$x_{n,\delta} = 2(E - \alpha A)x_{n-1,\delta} - (E - \alpha A)^2 x_{n-2,\delta} + \alpha^2 A y_\delta, \qquad x_{0,\delta} = x_{1,\delta} = 0;$$

$$(E + \alpha A)\,x_{n+1,\delta} = (E - \alpha A)\,x_{n,\delta} + 2\alpha y_\delta, \qquad x_{0,\delta} = 0;$$

$$x_{n+1,\delta} = x_{n,\delta} - \alpha\left(Ax_{n+1,\delta} - y_\delta\right), \qquad x_{0,\delta} = 0;$$

$$\left(E + \alpha^2 A^2\right)x_{n+1,\delta} = (E - \alpha A)^2 x_{n,\delta} + 2\alpha y_\delta, \qquad x_{0,\delta} = 0,$$

and investigated the *a priori* and *a posteriori* selection of the number of iterations for the case of an approximate right-hand side, for both an exact and approximate operator, and their convergence in the original Hilbert space norm. In the case of a non-unique solution, he proved convergence of these schemes to the minimum norm solution.