Journal of Computational and Applied Mathematics 300 (2016) 341–353

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Asymptotic behavior of varying discrete Jacobi–Sobolev orthogonal polynomials

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ARTICLE INFO

Article history:
Received 16 May 2015
Received in revised form 18 November 2015

MSC:
33C47
42C05

Keywords:
Sobolev orthogonal polynomials
Jacobi polynomials
Mehler–Heine formulae
Asymptotics
Zeros

ABSTRACT

In this contribution we deal with a varying discrete Sobolev inner product involving the Jacobi weight. Our aim is to study the asymptotic properties of the corresponding orthogonal polynomials and the behavior of their zeros. We are interested in Mehler–Heine type formulae because they describe the essential differences from the point of view of the asymptotic behavior between these Sobolev orthogonal polynomials and the Jacobi ones. Moreover, this asymptotic behavior provides an approximation of the zeros of the Sobolev polynomials in terms of the zeros of other well-known special functions. We generalize some results appeared in the literature very recently.

1. Introduction

One of the aims of this paper is the study of the asymptotic behavior of sequences of polynomials $\{Q_n^{(\alpha,\beta,M_n)}\}_{n\geq 0}$ orthogonal with respect to the inner product

$$(f,g)_{S,n} = \int_{-1}^{1} f(x)g(x)(1-x)^\alpha(1+x)^\beta dx + M_n f^{(j)}(1)g^{(j)}(1), \tag{1}$$

where $\alpha > -1$, $\beta > -1$, and $j \geq 0$.

We assume that $\{M_n\}_{n\geq 0}$ is a sequence of nonnegative real numbers satisfying

$$\lim_{n\to\infty} M_n n^\gamma = M > 0, \tag{2}$$

where γ is a fixed real number. Notice that this assumption is not very restrictive since the sequence $\{M_n\}_{n\geq 0}$ can behave asymptotically like any real power of the monomial n.

The main motivation to study this type of inner product arises from the papers [1,2]. In [1] the authors work with a measure supported on $[-1,1]$. However, in [2] the authors deal with measures supported on an unbounded interval. In both cases the authors consider measures with nonzero absolutely continuous part, i.e., they work with the so-called continuous

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http://dx.doi.org/10.1016/j.cam.2016.01.010
0377-0427/

Sobolev orthogonal polynomials. The main topic in those papers is how to balance the Sobolev inner product to equilibrate the influence of the two measures in the asymptotic behavior of the corresponding orthogonal polynomials. This inspires us to consider the discrete Sobolev inner product

$$(f,g)_S = \int fg d\mu_0 + M\int f^{(j)}g^{(j)}d\mu_1 = \int fg d\mu_0 + Mf^{(j)}(c)f^{(j)}(c),$$

which is a perturbation of a standard inner product. Now, making M dependent on n we can study the influence of the perturbation on the asymptotic behavior of the orthogonal polynomials. The literature on discrete Sobolev (or Sobolev-type) orthogonal polynomials is very wide, so we refer the interested readers on this topic to survey [3] and the references therein.

From here, in [4] the authors found the asymptotic behavior of a family of orthogonal polynomials with respect to a varying Sobolev inner product similar to (1), involving the Laguerre weight $w(x) = x^{\alpha}e^{-x}$, $\alpha > -1$. We remark that the techniques used in [4] are not useful in this case, and now we need to use more powerful techniques based on those considered in [5]. More recently, in [6] the same authors have even improved these techniques in such a way that they have obtained relevant results for the orthogonal polynomials with respect to a non-varying discrete Sobolev inner product being μ_0 a general measure.

Previously, in [7] J.J. Moreno-Balcázar obtained some results in this direction but only for the case $j = 0$. Again, the method used in that paper does not allow to tackle our problem.

We want to emphasize that our objective is to establish that the size of the sequence $\{M_n\}_{n\geq 0}$ has an essential influence on the asymptotic behavior of the orthogonal polynomials with respect to (1), but this influence is only local, that is, around the point where we have introduced the perturbation. In our case, this point is located at $x = 1$. Furthermore, we prove that this influence depends on the size of the sequence $\{M_n\}_{n\geq 0}$ and its relation with the parameter α in the Jacobi weight and the order of the derivative in (1). It is important to remark that for a sequence $\{M_n\}_{n\geq 0}$, we have a sequence of orthogonal polynomials for each n, so we have a square tableau $\{Q_k^{(\alpha,\beta,M_n)}\}_{k\geq 0}$. Here, we deal with the diagonal of this tableau, i.e. $\{Q_n^{(\alpha,\beta,M_n)}\}_{n\geq 0} = \{Q_0^{(\alpha,\beta,M_0)}(x), Q_1^{(\alpha,\beta,M_1)}(x), \ldots, Q_i^{(\alpha,\beta,M_i)}(x), \ldots\}$. At this point, in order to simplify the notation, we will denote $Q_n^{(\alpha,\beta,M_n)}(x) = Q_n(x)$.

A second aim of this paper is to establish a simple asymptotic relation between the zeros of the Sobolev polynomials which are orthogonal with respect to (1) and the zeros of combinations of Bessel functions of the first kind. This relation is deduced as an immediate consequence of Mehler–Heine formulae (Theorem 2) and they have a numerical interest since we provide an estimate of the zeros of these polynomials.

Since Jacobi classical orthogonal polynomials are involved in the varying inner product (1), we recall some of their basic properties. Jacobi polynomials are orthogonal with respect to the standard inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1.$$

In the sequel, we will work with the sequence $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$, $\alpha > -1$ and $\beta > -1$, normalized by (see [8, f. (4.1.1)])

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}. \tag{3}$$

The derivatives of Jacobi polynomials satisfy (see, [8, f. (4.21.7)])

$$(P_n^{(\alpha,\beta)}(x))^{(k)} = \frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}P_{n-k}^{(\alpha+k,\beta+k)}(x), \quad k \geq 0. \tag{4}$$

Using (3) and (4), we deduce

$$(P_n^{(\alpha,\beta)}(1))^{(k)} = \frac{1}{2^k}\frac{\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+1)}{\Gamma(n-k+1)\Gamma(\alpha+k+1)}, \tag{5}$$

where $(P_n^{(\alpha,\beta)}(1))^{(k)}$ denotes the kth derivative of $P_n^{(\alpha,\beta)}$ evaluated at $x = 1$.

We also note that the squared norm of a Jacobi polynomial is (see, [8, f. (4.3.3)]):

$$\|P_n^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{6}$$

Finally, we will use the Mehler–Heine formula for classical Jacobi polynomials

Theorem 1 *([8, Th. 8.1.1]).* *Let $\alpha, \beta > -1$. Then,*

$$\lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}\left(\cos\left(\frac{x}{n}\right)\right) = \lim_{n\to\infty}\frac{1}{n^{\alpha}}P_n^{(\alpha,\beta)}\left(1-\frac{x^2}{2n^2}\right) = (x/2)^{-\alpha}J_{\alpha}(x),$$