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A Chebyshev collocation method for a class of Fredholm integral equations with highly oscillatory kernels[☆]

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ABSTRACT

Based on the Filon–Clenshaw–Curtis method for highly oscillatory integrals, and together with the Sommariva's result (Sommariva, 2013) for Clenshaw–Curtis quadrature rule, we present a Chebyshev collocation method for a class of Fredholm integral equations with highly oscillatory kernels, whose unknown function is assumed to be less oscillatory than the kernel. In the proposed method, the Filon–Clenshaw–Curtis method is used to compute the involved oscillatory integrals, which makes the proposed method very precise. By solving only a small system of linear equations, we can obtain a very satisfactory numerical solution. The performance of the presented method is illustrated by several numerical examples. Compared with the method proposed by Li et al. (2012), this method enjoys a lower computational complexity. Furthermore, numerical examples show that the presented method has a competitive advantage on the accuracy compared with the method in Li et al. (2012).

1. Introduction

In the last few years, there are many substantial efforts towards highly oscillatory integrals. There exist many efficient methods for highly oscillatory integrals, such as asymptotic method [1–3], Filon-type method [4–13], Levin-type method [14–18], steepest descent method [19–26], etc. Especially, the Filon–Clenshaw–Curtis (FCC) method is one of the most popular methods for the oscillatory integral $\int_{-1}^{1} f(x)S(\omega,x)dx$, where $S(\omega,x)$ is a highly oscillatory function for a large frequency ω. The FCC method is defined by replacing $f(x)$ with its interpolation polynomial $p_N(x) = \sum_{j=0}^{N} a_j T_j(x)$ at $N+1$ Clenshaw–Curtis points $\{\cos(\frac{j\pi}{N})\}_{j=0}^{N}$ in terms of Chebyshev polynomials of the first kind,

$$\int_{-1}^{1} f(x)S(\omega,x)dx \approx \int_{-1}^{1} p_N(x)S(\omega,x)dx = \sum_{j=0}^{N} a_j\tilde{M}_j(\omega),$$

where $T_j(x)$ is the Chebyshev polynomial of the first kind of degree j, and $\tilde{M}_j(\omega) = \int_{-1}^{1} T_j(x)S(\omega,x)dx$ for $j = 0, 1, \ldots, N$ are modified moments that can be computed by either a recursive formula or Oliver's algorithm [27] usually. The coefficients

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a_j can be computed fast by the Fast Fourier Transform (FFT) [28]. In the current paper, we extend the FCC method for highly oscillatory integrals to the integral equations with highly oscillatory kernels.

The integral equations with highly oscillatory kernels occur in a number of application areas, for instance in electromagnetics, acoustic scattering, laser engineering and quantum mechanism. Their numerical solutions have attracted much attention during the last few years. In 2007 and 2010, Brunner et al. gave two comprehensive surveys and proposed some open problems [29,30] on the numerical solutions of Volterra integral equations with highly oscillatory kernels. Numerical methods for Volterra integral equations with highly oscillatory kernels are discussed in [31–34,13,35,36]. Fredholm integral equations with highly oscillatory kernels are considered in [37–39]. Brunner et al. studied the spectral problem for a class of highly oscillatory Fredholm integral operators in [37]. Ursell considered the asymptotic properties of the solution for a second kind of Fredholm integral equation with a highly oscillatory kernel $f(x, y)e^{i\omega|x-y|}$ in [39]. Recently, based on an improved-Levin method [17] for the highly oscillatory integral, Li et al. showed an efficient collocation method for a rapid solution of 1D Fredholm oscillatory integral equation in [38]

$$\phi(x) - \int_a^b f(x, y)e^{i\omega g(x,y)}\phi(y)dy = h(x), \quad \omega \gg 1, \ x \in [a, b], \tag{1.1}$$

where $f(x, y)$, $g(x, y)$ and $h(x)$ are given functions, and the unknown function $\phi(x)$ is assumed much less oscillatory than the kernel function $f(x, y)e^{i\omega g(x,y)}$. In this paper, we refer to the method in [38] as Li's method for convenience. In what follows, we give a short description of Li's method for the case of free of stationary phase points.

Let $y_k = \frac{b+a}{2} + \frac{b-a}{2}\cos(\frac{k\pi}{N})$ for $k = 0, 1, \ldots, N$ be the $(N+1)$ Clenshaw–Curtis points on the interval $[a, b]$. The function values of $\phi(x)$ on the Clenshaw–Curtis nodes $[\phi(y_0), \phi(y_N), \ldots, \phi(y_N)]^T$ are approximated by solving the system of linear equations

$$(\mathbf{I} - \mathbf{U})\tilde{\mathbf{\Phi}} = \tilde{\mathbf{H}}, \tag{1.2}$$

where $\mathbf{I}$ is the $(N+1) \times (N+1)$ identity matrix, $\tilde{\mathbf{\Phi}} = \left[\tilde{\phi}_0, \tilde{\phi}_1, \ldots, \tilde{\phi}_N\right]^T$, and

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_N \end{bmatrix} \in \mathbb{C}^{(N+1)\times(N+1)}, \qquad \tilde{\mathbf{H}} = \begin{bmatrix} h(y_0) \\ h(y_1) \\ \vdots \\ h(y_N) \end{bmatrix} \in \mathbb{C}^{(N+1)\times 1}. \tag{1.3}$$

Each $\mathbf{U}_j \in \mathbb{C}^{1\times(N+1)}$ is a row vector which is given by

$$\mathbf{U}_j = \mathbf{Q}_j \left(\frac{2}{b-a}\mathbf{D} + i\omega\mathbf{\Sigma}_j\right)^{-1} \mathrm{diag}(\mathbf{F}_j), \quad j = 0, 1, \ldots, N, \tag{1.4}$$

where $\mathbf{D}$ is the Chebyshev differentiation matrix [40], and

- $\mathbf{Q}_j = [e^{i\omega g(y_j,1)}, 0, \ldots, 0, -e^{i\omega g(y_j,-1)}]$ is a vector with the zero entries except the first and last entries.
- $\mathbf{\Sigma}_j = \mathrm{diag}\left(g'(y_j, y_0), g'(y_j, y_1), \ldots, g'(y_j, y_N)\right)$ is a diagonal matrix.
- $\mathbf{F}_j = \left[f(y_j, y_0), f(y_j, y_0), \ldots, f(y_j, y_N)\right]^T$ is a numerical vector composed of different function values.

There is a little nonessential change on $\mathbf{U}_j$ for the case of stationary points, but the computational complexity is the same. For more details of this case, we refer the readers to [38].

Form (1.2)–(1.4), we can see that Li's method requires $O\left((N+1)^4\right)$ operations for computing the entries of $\mathbf{U}$ and $O\left((N+1)^3\right)$ operations for solving the linear system (1.2) by using a direct solver (such as the Gauss elimination method or the LU factorization method). Even employing iterative solvers (such as the GMRES method), the computational complexity for obtaining $\mathbf{U}$ is still $O\left((N+1)^3\right)$ and $O\left((N+1)^2\right)$ for solving Eq. (1.2).

Since the integral ranging over $[a, b]$ can be transformed to the interval $[-1, 1]$. Thus, for convenience, we use the interval $[-1, 1]$ instead of $[a, b]$ in the integral equation (1.1) in the following. In this work, we consider a different Chebyshev collocation method for a class of Fredholm equations of the form

$$\phi(x) - \int_{-1}^1 f(x, y)e^{i\omega g(x,y)}\phi(y)dy = h(x), \quad \omega \gg 1, \ x \in [-1, 1], \tag{1.5}$$

but only considering three cases for $g(x, y)$

$$g(x, y) = xy, \qquad g(x, y) = |x - y| \quad \text{and} \quad g(x, y) = (x - y)^2, \tag{1.6}$$

on account of an important application in laser theory [37] for these three situations. For a more general function $g(x, y)$, the method presented in this paper is still valid if the modified moments $\int_{-1}^1 T_k(y)e^{i\omega g(x,y)}dy$ can be computed fast. It is showed in [39] that the solution of a Fredholm integral equation of 2nd kind with highly oscillatory kernel also oscillates rapidly in