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Asymptotic stochastic dominance rules for sums of i.i.d. random variables

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ABSTRACT

In this paper, we deal with stochastic dominance rules under the assumption that the random variables are stable distributed. The stable Paretian distribution is generally used to model a wide range of phenomena. In particular, its use in several applicative areas is mainly justified by the generalized central limit theorem, which states that the sum of a number of i.i.d. random variables with heavy tailed distributions tends to a stable Paretian distribution. We show that the asymptotic behavior of the tails is fundamental for establishing a dominance in the stable Paretian case. Moreover, we introduce a new weak stochastic order of dispersion, aimed at evaluating whether a random variable is more "risky" than another under condition of maximum uncertainty, and a stochastic order of asymmetry, aimed at evaluating whether a random variable is more or less asymmetric than another. The theoretical results are confirmed by a financial application of the obtained dominance rules. The empirical analysis shows that the weak order of risk introduced in this paper is generally a good indicator for the second order stochastic dominance.

1. Introduction

Historically, the Gaussian distribution has always been considered the most relevant probability distribution, due to its important role in statistical inference and to its use in approximating real-valued random variables (r.v.) in many fields of study.

The key reason why the normal distribution is so important is because of the Central Limit Theorem, which states that the sum of a sufficiently large number of independent and identically distributed (i.i.d.) random variables is approximately normally distributed (regardless of the underlying distribution) provided that each random variable has finite variance. When we deal with random variables which do not have finite variance, and thereby when the Gaussian distribution is unsuitable as a limit distribution, we rely on the generalized central limit theorem [1], which states that the sum of a number of i.i.d. random variables with a Paretian tail distribution (decreasing as $|x|^{-\alpha-1}$, where $0 < \alpha < 2$, and therefore having infinite variance) tends to a stable Paretian distribution as the number of summands grows.

Nevertheless, in real world problems the assumption of finite variance is not always appropriate, because several phenomena are well described by random variables which are generally not square-integrable. Therefore, heavy tailed and skewed distributions have often been considered a more realistic distributional assumption for a wide range of natural and

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man-made phenomena, such as natural sciences (see e.g. [2]), social sciences (see e.g. [3]) or econometrics (see [4] and the references therein). For this reason, the stable Paretian distribution has been proposed as an alternative model to the Gaussian distribution in many different frameworks.

In the financial literature, it is well known that asset returns are not normally distributed, as several studies by Mandelbrot (see [5–8]) and Fama (see [9–11]) recognized an excess of kurtosis and non-zero skewness in the empirical distributions of financial assets, which often lead to the rejection of the assumption of normality and proposition of the stable Paretian distribution as an alternative model for asset returns. The Fama and Mandelbrot's conjecture was supported by numerous empirical investigations in the subsequent years, (see [12,13]).

In view of the several financial applications of the stable Paretian distribution, the aim of this paper is to (stochastically) order stable distributed random variables. Stochastic dominance rules quantifies the concept of one random variable being "preferable" to another, by establishing a partial order in the space of distribution functions. For instance, in a financial context, stochastic orderings are used to establish an order of preferences for investors whose utility functions share certain characteristics [14]. Indeed, it is well known that stochastic dominance rules are generally aimed at addressing investors and institutions towards the best choices in terms of expected gain and risk (see, among others, [15–19]).

Actually, the financial interpretation of stochastic dominance is straightforward, when the order of preferences can be summarized by maximum expected gain and minimum risk. According to the literature [20], for the expected gain we generally use the first moment (expectation) and for the risk we generally use the variance: this is especially suitable in case of normality. To justify the choices based on the so-called *mean–variance* rule, we need that the return distributions are elliptical and asymptotically approximated by a Gaussian law. Under these assumptions, the mean–variance rule is consistent with the choices of non-satiable and risk averse investors.

However, in this paper, we do not rely on these non-realistic assumptions. In fact, we study the conditions for ordering choices of non satiable risk averse investors in the general case that (i) the r.v. does not necessarily have finite variance; (ii) the distribution is asymptotically approximated by a stable Paretian law. It is well known in literature that we can obtain the second order stochastic dominance between stable distributions by a mean-dispersion comparison (similar to the Gaussian case), but only when the stable distributions present the same skewness parameter and index of stability [12,21]. In this paper, we present more general results, by considering two fundamental aspects of the stable Paretian distributions, namely: the tail behavior and the asymmetry. In particular, we show that the tail parameter (i.e. index of stability) is crucial for establishing a dominance, in that a distribution with heavier tails cannot dominate (with respect to the discussed preference order) a distribution with lighter tails. Moreover, we define a new stochastic order, weaker then the second order stochastic dominance, and prove that it holds if some conditions on the skewness parameters are verified.

In Section 2, we also introduce a stochastic order of asymmetry which is based on the absolute moments of appropriately standardized random variables. This definition generalizes the traditional definition of asymmetry, based on the Pearson's moment coefficient, and is especially suitable for dealing with heavy tailed r.v.'s, whose moments of order 2 (and 3) do not exist finite. In Section 3 we prove that the skewness parameter of the stable Paretian distribution is indeed coherent with the stochastic order of asymmetry. Finally, in Section 4 we analyze the empirical distributions of a set of asset returns from the US equity market, and show the validity and usefulness of our theoretical results.

2. Stochastic dominance rules for dispersion and asymmetry order

In this section, we provide some general results which hold for any kind of random variable. In particular, we introduce new stochastic orderings which will be useful when dealing with stable distributions. We first recall the definitions of some classical stochastic orders.

Definition 1. – First order stochastic dominance (FSD): we say that X dominates Y with respect to the first stochastic dominance order (in symbols X FSD Y) if and only if $F_X(t) \leq F_Y(t)$, $\forall t \in \mathbb{R}$, or, equivalently X FSD Y if and only if $E(g(X)) \geq E(g(Y))$ for any increasing function g.

– Second order stochastic dominance (SSD or increasing concave order): we say that X dominates Y with respect to the second stochastic dominance order (in symbols X SSD Y or $X \geq_{icv} Y$) if and only if $\int_{-\infty}^{t} F_X(u)\,du \leq \int_{-\infty}^{t} F_Y(u)\,du$, $\forall t \in \mathbb{R}$ or, equivalently X SSD Y if and only if $E(g(X)) \geq E(g(Y))$ for any increasing and concave function g. Obviously X FSD Y implies also X SSD Y.

– Increasing and convex order: we say that X dominates Y with respect to the increasing convex order (in symbols $X \geq_{icx} Y$) if and only if $\int_{t}^{+\infty} 1 - F_X(u)\,du \geq \int_{t}^{+\infty} 1 - F_Y(u)\,du$, $\forall t \in \mathbb{R}$ or, equivalently $X \geq_{icx} Y$ if and only if $E(g(X)) \geq E(g(Y))$ for any increasing and convex function g. Obviously X FSD Y implies also $X \geq_{icx} Y$.

– Rothschild–Stiglitz stochastic dominance (RS or concave order): we say that X dominates Y with respect to the Rothschild–Stiglitz order (in symbols X RSY) when X SSD Y and $E(X) = E(Y)$ (or $Y \geq_{icx} X$ and $E(X) = E(Y)$) or equivalently X RS Y if and only if $E(g(X)) \geq E(g(Y))$ for any concave function g.

The following theorem extends the well-known result of Hanoch and Levy [22] which determines a sufficient condition for SSD. Similarly, we obtain a sufficient condition which allows to deny the SSD and the $\geq_{icx}$ dominance. Both conditions are based on the number of crossing points between distributions, and will be really useful in Section 3. In particular, we establish that if two distributions have an even number of crossing points, then the SSD and the $\geq_{icx}$ ordering cannot hold.