Journal of Computational and Applied Mathematics 305 (2016) 18–35

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A heuristic and evolutionary algorithm to optimize the coefficients of curve parametrizations

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ARTICLE INFO

Article history:
Received 13 August 2015
Received in revised form 20 March 2016

Keywords:
Rational curve
Parametrization height
Heuristic algorithm
Evolutionary algorithm
Arithmetic optimality

ABSTRACT

Parametric representations may have unnecessarily huge integer coefficients. This can be a computational problem in practical applications. In this paper we present an evolutionary algorithm that reduces the maximum length of the coefficients for a proper curve parametrization with integer coefficients. This method is tested with different families of parametrizations, and as we show the results are very satisfactory in terms of achievable quality and runtime consumption. According to our knowledge, this is the first algorithmic approach to this problem.

1. Introduction

Rational algebraic curves and surfaces are basic tools in computer graphics, CAD/CAM, and surface/geometric modeling (see e.g. [1]) and they are applicable in many mathematical areas as, for instance, diophantine equations (see e.g. introduction in [2]) or symbolic solutions of algebraic differential equations (see e.g. [3–5]). One of the main advantages of these geometric objects is that they can be represented parametrically by means of rational functions. Nevertheless, since rational algebraic sets admit infinitely many different rational parametrizations, the power of their applicability varies depending on which parametrization is chosen.

In order to improve the applicability of the parametric representations, several authors have addressed the problem of transforming a given parametrization into a new parametrization satisfying certain required optimality criterion as, for instance, the injectivity or the surjectivity of the induced rational map, the degree (over the ground field) of the field of parametrization, or the height, i.e., the maximal absolute value of the integer coefficients of the parametrization (see e.g. [2]). In this paper we put the main focus on curves; readers interested in the analysis of surfaces may consult e.g. [6–14]. Injective parametrizations are usually called proper parametrizations and, as a consequence of Lüroth's theorem, any rational curve can always be parametrized properly. Algorithmically, the problem was solved in [15]. Similarly, any rational curve can always be parametrized by means of a surjective parametrization (see [16,10], or [17] for algorithmic approaches), although one may need to introduce complex numbers in the description of the parametrization. Concerning the field of parametrization, Hilbert and Hurwitz [18] stated that one can always parametrize over a field algebraic extension of degree at most two over the ground field. Algorithms to achieve such parametrizations can be found in [19–21]. Nevertheless, computing parametrizations with optimal height is, up to our knowledge, a fully open problem; in [22] one can see an example of applicability of our results. We refer to this last question as the arithmetic optimality problem, and in this paper we explain our approach to solve it by means of evolutionary computation.

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http://dx.doi.org/10.1016/j.cam.2016.03.020

Our proposal in this paper is to approach the problem by means of evolutionary computation improving the initial results and ideas given in [23]. Evolution strategies have been developed since the 1960s as explained and analyzed in detail in, e.g., [24–26]. Since then, evolutionary computation has been applied to solve problems in many different areas (see, e.g., [27,28]). Nevertheless, this does not mean that any optimization problem can be directly approached via evolutionary techniques; a pre-analysis of a suitable search strategy is required, since otherwise the evolutionary search may turn out to be a blind random process.

Let us briefly describe why our evolutionary strategy is suitable for the arithmetic optimality problem. We look for different changes of parameters such that the input proper parametrization is transformed into a better one; i.e., here, one with smaller height. We know that all possible changes of parameters are of the form

$$\frac{at+b}{ct+d} \quad \text{with } a, b, c, d \in \mathbb{Z} \text{ and } ad - cb \neq 0. \tag{1}$$

We visualize the space of solution candidates in 3D in so-called fitness landscapes [29]: In the x-axis we set the numerator (namely the pair (a, b)), in the y-axis we set the denominator (i.e. the pair (c, d)) and the color represents the height of the resulting parametrization; red is small height (i.e. good result) and blue is big height (i.e. bad result). Such a direct description of the search space is not suited for the evolutionary algorithm since it is not smooth (see Fig. 5 (left)) and small variations (for instance, due to mutations) will produce big changes of quality in the answer. Instead, in Section 3, we prove that one can associate to each numerator (or denominator, resp.) a quantity that partially indicates the quality of the final answer (see Eq. (27)). Thus, before generating the x-axis and the y-axis of the fitness landscape, we order the numerators as well as the denominators according to this partial quality. This produces fitness landscapes (see Fig. 5 (right)) which are well suited for our purposes. In this situation, in each region of the search space we look for improvement by optionally using strict offspring selection.

Based on these ideas, in this paper we present an evolutionary algorithm for dealing with the arithmetic optimality problem with a very satisfactory performance. The paper is structured as follows: In Section 2 we formally state the problem we aim to solve in the paper. Section 3 has 4 subsections: In Section 3.1 we analyze the search space, in Section 3.2 we study the notion of partial and complete quality, Section 3.3 is devoted to the evolutionary strategy, and in Section 3.4 we present our solution approach in detail. In the last section we empirically analyze the algorithm's performance: we use it for optimizing coefficients of parametrizations associated with a random family of conics; for a random family of parametrizations; as well as of three different families of space curves where the height of the applied Möbius transformation varies, and in all cases the results are very satisfactory. We also demonstrate the algorithm's performance used with varied algorithmic parameter settings, the details of the test results are given in tables that are shown in the paper's Appendix.

2. Problem statement

In this section, we introduce the notation and terminology that will be used throughout the paper, and we state the problem we will deal with.

Let $\mathbf{P}(t)$ be a proper rational parametrization (with integer coefficients) of a curve $\mathbf{C}$ in the r-dimensional space $\mathbb{R}^r$. We recall that proper means that the parametrization is injective almost everywhere, i.e., almost all points in $\mathbf{C}$ are reached by exactly one parameter value via $\mathbf{P}(t)$ (see e.g. [2]). Let us **assume** that $\mathbf{P}(t)$ is expressed as

$$\mathbf{P}(t) = \left(\frac{p_1(t)}{q(t)}, \dots, \frac{p_r(t)}{q(t)}\right) \tag{2}$$

such that

1. p_i, q are polynomials with integer coefficients such that $\gcd(p_1, \dots, p_r, q) = 1$,
2. no component of $\mathbf{P}(t)$ is constant.

It is clear that assumption (1) is always reachable. Below, we will see that condition (2) does not imply either any loss of generality.

We recall that the height of a polynomial $f(t) = a_0 + \cdots + a_m t^m \in \mathbb{R}[t]$ is defined as $\mathrm{H}(f) = \max\{|a_0|, \dots, |a_m|\}$. We define the height of the parametrization $\mathbf{P}(t)$ as

$$\mathrm{H}(\mathbf{P}(t)) = \max\{\mathrm{H}(p_1), \dots, \mathrm{H}(p_r), \mathrm{H}(q)\}. \tag{3}$$

In this situation, the arithmetic optimality problem can be stated as follows:

> Problem Statement (General Version). Given $\mathbf{P}(t)$ as above, determine a rational change of parameter $\phi(t)$ such that $\mathbf{Q}(t) := \mathbf{P}(\phi(t))$ is proper, has integer coefficients, and $\mathrm{H}(\mathbf{Q}(t))$ is minimal.

We observe that if any component (say the first) of $\mathbf{P}(t)$ is a constant $c \in \mathbb{R}$, also for any other parametrization of $\mathbf{C}$ the first component equals c. Therefore, assumption (2) does not imply loss of generality.

Since $\mathbf{P}$ is proper, we have $\phi(t) = \mathbf{P}^{-1}(\mathbf{Q}(t))$. Moreover, since elimination theory techniques do not extend the ground field, $\mathbf{P}^{-1}$ has integer coefficients, and hence $\phi(t) \in \mathbb{Z}(t)$. Note that the properness condition is fundamental for the previous statement: for instance, $\mathbf{P}(t) = (\frac{1}{2}t^2, \frac{1}{2}t^2)$ is a non-proper parametrization of the line $y = x$, and $\mathbf{P}(\sqrt{2}t) = (t^2, t^2)$.