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Mean square stability of two classes of theta method for neutral stochastic differential delay equations☆

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ABSTRACT

In this paper, a stochastic linear theta (SLT) method is introduced and analyzed for neutral stochastic differential delay equations (NSDDEs). We give some conditions on neutral item, drift and diffusion coefficients, which admit that the diffusion coefficient can be highly nonlinear and does not necessarily satisfy a linear growth or global Lipschitz condition. It is proved that, for all positive stepsizes, the SLT method with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean stable and so is $\theta \in [0, \frac{1}{2})$ under a stronger assumption. Furthermore, we consider the split-step theta (SST) method and obtain a similar but better result. That is, the SST method with $\theta \in [\frac{1}{2}, 1]$ is exponentially mean stable and so is $\theta \in [0, \frac{1}{2})$. Finally, two numerical examples are given to show the efficiency of the obtained results.

1. Introduction

As is well known, there have appeared a large number of works on neutral stochastic differential delay equations (NSDDEs) (see [1–5]) since they have been widely applied to many fields such as economics, finance, physics, biology, medicine, and other sciences. The stability issue of NSDDEs is one of the most important problems in their research field. Recently, various stability theorems of stochastic differential systems, for example, moment stability (M-stability, see [6,7]) and almost sure stability (or the trajectory stability (T-stability), see [8]), have been reputed in the literature. Some of the stability criteria related neutral stochastic functional differential equations (NSFDEs) were considered in [9–11,2,12,4,5] and the references therein. On the other hand, many NSDDEs may not have explicit solutions. Therefore, it seems to be interesting and necessary to study the numerical solutions of NSDDEs (see [13–18]). However, there have been very few works to consider the theta methods on NSDDEs, despite its practical importance and extensive application.

Luckily, there have appeared some results on the numerical solutions about theta methods of stochastic ordinary differential equations (SODEs). Stochastic linear theta (SLT) method is the simplest method, and it has been widely used in the literature. For example, the mean square stability of the SLT method was investigated in [19,8,20–22] for linear SODEs and in [23] for nonlinear SODEs. For stochastic differential delay equations (SDDEs), Huang [24] investigated the expo-

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nential mean square stability of SLT method and so was Mao in [25]. Zong et al. in [26] proved that the SLT method can inherit the exponential mean square stability of the exact solution for SODEs and SDDEs. Besides, Huang also introduced another theta method called the split-step theta (SST) method in [27]. For the special case of $\theta = 0$, this approximation is EM approximation, and for the case of $\theta = 1$, this approximation is equivalent to the split-step backward Euler (SSBE) method. Both for SLT and SST methods, Huang in [26] revealed that the linear growth condition on the drift coefficient is necessary with $\theta \in [0, \frac{1}{2})$ to be mean square stable, but for $\theta \in [\frac{1}{2}, 1]$, two methods can reproduce the exponential mean square stability without the linear growth condition. Also, Liu et al. [28] studied the mean-square stability of the stochastic theta method for linear scalar model equations. Baker and Buckwar [29] analyzed the exponential stability in pth moment of the stochastic theta method by using the Halanay inequality. Wang and Gan [30] investigated the mean-square exponential stability of a split-step Euler method. However, all of the above results are derived from SDDEs in which the diffusion coefficient needs to satisfy a linear growth or global Lipschitz condition. Moreover, these results ignored the effect of the neutral term, which often yields much difficulty.

Motivated by the above discussion, in this paper, we study the stability of numerical methods for NSDDEs under some conditions on the drift coefficient, diffusion coefficient and neutral term. These conditions admit that the diffusion coefficient is highly nonlinear, and it does not necessarily satisfy the linear growth or global Lipschitz condition. To the best of our knowledge, there is only one paper [31] studying the stability of SST and SLT methods for NSDDES. However, in this paper we propose some weaker assumptions on the drift and diffusion coefficients than those in [31]. Indeed, we do not require the condition that f and g need to satisfy the global Lipschitz condition. Moreover, the SLT and SST methods presented in this paper generalize and improve those given in [31]. In the paper, we prove that, for all positive stepsizes, the SLT method with $\theta \in [\frac{1}{2}, 1]$ is asymptotically mean square stable and so is $\theta \in [0, \frac{1}{2})$ under a stronger assumption. Furthermore, we also establish the SST method with $\theta \in [\frac{1}{2}, 1]$ is exponentially mean stable and so is $\theta \in [0, \frac{1}{2})$. Hence, we can see that the SST method has a better exponential stability property than the SLT method.

The rest of the paper is arranged as follows. In Section 2, we introduce some notations, assumptions and preliminary lemmas. In Section 3, we use the SLT method to discuss the mean square stability of numerical solutions to NSDDEs. In Section 4, we use the SST method to investigate the mean square stability of numerical solutions to NSDDEs. After some numerical examples are provided to illustrate the obtained results in Section 5, we conclude the paper with some general remarks in Section 6.

2. Notations, assumptions and lemmas

Throughout this paper, we use the following notations. If A is a vector or matrix, its transpose is denoted by A^T. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace norm in $\mathbb{R}^{n\times d}$(denoted by $|A| = \sqrt{trace(A^TA)}$). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\{w(t), t \geq 0\}$ be a d-dimensional Brownian motion defined on the probability space.

Let $D : \mathbb{R}^n \longmapsto \mathbb{R}^n, f : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{R}^n, g : \mathbb{R}^n \times \mathbb{R}^n \longmapsto \mathbb{R}^{n\times d}$ be Borel measurable functions. Let us consider the following neutral stochastic differential delay equation

$$d[y(t) - D(y(t-\tau))] = f(y(t), y(t-\tau))dt + g(y(t), y(t-\tau))dw(t), \quad t > 0, \tag{2.1}$$

with initial data $y(t) = \Phi(t) \in C([-\tau, 0]; \mathbb{R}^n)$ satisfying

$$\sup_{-\tau\leq t\leq 0} \mathbb{E}[\Phi^T(t)\Phi(t)] < +\infty. \tag{2.2}$$

For the purpose of stability, assume that $D(0) = f(0, 0) = 0, g(0, 0) = 0$. This implies that system (2.1) admits a trivial solution.

There exist many numerical schemes for stochastic differential equations in the literature. If an appropriate interpolation procedure for the delay argument is employed, these schemes can be adapted to solve NSDDEs. An adaptation of the classic stochastic theta method to system (2.1) leads to

$$y_{n+1} = y_n + D(\bar{y}_{n+1}) - D(\bar{y}_n) + \theta\Delta f(y_{n+1}, \bar{y}_{n+1}) + (1-\theta)\Delta f(y_n, \bar{y}_n) + g(y_n, \bar{y}_n)\Delta w_n, \tag{2.3}$$

where $\Delta > 0$ is the time stepsize, y_n is an approximation to $y(t_n)$, $\theta \in [0, 1]$ is a fixed parameter, $\Delta w_n = w(t_{n+1}) - w(t_n)$, and $\bar{y}_n$ denotes an approximation to the delay argument $y(t_n - \tau)$.

For an arbitrarily fixed time stepsize Δ, there exist a unique positive integer m and a real number $\delta \in [0, 1)$ such that $\tau = (m-\delta)\Delta$. This implies that $y(t_n - \tau) = y(t_{n-m} + \delta\Delta)$. Therefore, it is natural to define y_n by the linear interpolation

$$\bar{y}_n = \delta y_{n-m+1} + (1-\delta)y_{n-m}, \tag{2.4}$$

where $\bar{y}_n = \Phi(t_n)$ for $n \leq 0$.

In order to distinguish this method and another method with parameter θ below, we will refer to (2.3) as the stochastic linear theta (SLT) method following the notation in [27]. An adaptation of the split-step theta (SST) method in [27] to