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A weak Galerkin generalized multiscale finite element method

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ABSTRACT

In this paper, we propose a general framework for weak Galerkin generalized multiscale (WG-GMS) finite element method for the elliptic problems with rapidly oscillating or high contrast coefficients. This general WG-GMS method features in high order accuracy on general meshes and can work with multiscale basis derived by different numerical schemes. A special case is studied under this WG-GMS framework in which the multiscale basis functions are obtained by solving local problem with the weak Galerkin finite element method. Convergence analysis and numerical experiments are obtained for the special case.

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1. Introduction

Many problems arising from science and engineering have features at multiple scales such as physical processes in strongly heterogeneous media including deformation or diffusion in composite materials and flow in porous medium. For the problems with multiple scales and high contrast, the convergence of the standard finite element methods requires a mesh size h small enough to resolve the fine scale size. The cost of computations is often prohibitively expensive. Multiscale finite element methods are designed to solve multiscale problems efficiently that have the dimension of the coarse grid and resolve the fine scale features through construction of multiscale basis. The development of multiscale finite element methods is an active research field with significant study over the past decade. Multiscale techniques have been applied to different finite element methods for solving elliptic problems, for continuous finite element methods [1–7], for mixed finite element methods [8–11], for discontinuous Galerkin methods [12].

The studies in [13,5] show that the accuracy of multiscale finite element methods is sensitive to the boundary conditions imposed on computing basis functions. The imposed boundary conditions should have similar oscillatory behavior as the fine-scale solution. Taking this into consideration, generalized multiscale finite element methods construct additional multiscale basis functions by solving a carefully designed local spectral problem to enrich the initial multiscale space [14–18].

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Specially, the generalized multiscale finite element method works well for more general problems such as materials with non-periodic properties, non-separable scales, and random coefficients.

Weak Galerkin method is a newly developed general finite element technique for solving partial differential equations (PDE). The WG methods, by design, make use of discontinuous piecewise polynomials and can be considered as a natural extension of the standard Galerkin finite element methods for functions with discontinuity. Therefore the WG method has the flexibility of using discontinuous elements and the simplicity in formulation of using continuous elements. In summary, the WG methods have the following features:

1. The WG method is highly flexible in which high order approximations and finite element partitions with arbitrary shape of polygons and polyhedrons can be used.
2. The WG method has simple formulation which implies easy implementation and easy error analysis. The WG formulation can be derived by replacing standard derivatives by weakly-defined derivatives in the corresponding variational forms of the PDEs, with an option of adding a parameter independent stabilizer.

The key components of the WG methods are weak derivatives and parameter free stabilizers which are introduced to enforce the connection of discontinuous approximations between elements. The definitions of weak derivatives in the WG methods can be very general which should depend on the natures of the approximation functions. The general definition of weak derivative makes the WG method widely applicable and easy to fit into the frameworks of many numerical techniques such as multiscale finite element methods and least-squares finite element methods.

The weak Galerkin method was first introduced in [19]. Since then the WG methods have been successfully applied for solving the second order elliptic problems in [20–26], for the biharmonic equations in [27–29] and for the Brinkman equations [30]. The WG methods have also been used for the elliptic interface problems [31], for the Helmholtz equation [32, 33], for the Maxwell equations [34] and for the Stokes equations [35].

In this paper, we will develop a weak Galerkin generalized multiscale finite element framework for the elliptic problem with rapidly oscillating and high contrast coefficients. Weakly defined derivative makes the WG-GMS methods compatible with multiscale basis derived by different numerical schemes such as finite element methods, continuous or discontinuous; spectral methods or others. A special case under the WG-GMS framework is investigated. In this case study, the weak Galerkin finite element method is used to compute local snapshots. The numerical results demonstrate the accuracy of the WG-GMS finite element solutions after significant dimension reductions.

We consider the second order elliptic equation with rapidly oscillating or high contrast coefficients that seeks an unknown function u satisfying

$$-\nabla\cdot(a\nabla u) = f, \quad \text{in } \Omega, \tag{1.1}$$

$$u = g, \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polytopal domain in $\mathbb{R}^d$ (polygonal or polyhedral domain for $d = 2, 3$), ∇u denotes the gradient of the function u, and a is a symmetric $d \times d$ matrix-valued function in Ω. We shall assume that there exists a positive number $\lambda > 0$ such that

$$\xi^t a\xi \geq \lambda\xi^t\xi, \quad \forall\xi \in \mathbb{R}^d. \tag{1.3}$$

Here ξ is understood as a column vector and ξ^t is the transpose of ξ.

2. Framework of the WG-GMS finite element method

Let $\mathcal{T}_H$ be a partition of a domain Ω consisting of polygons in two dimensions or polyhedra in three dimensions satisfying a set of conditions specified in [21]. Denote by $\mathcal{E}_H$ the set of all edges or flat faces in $\mathcal{T}_H$. Let $\mathcal{E}_H^0 = \mathcal{E}_H \setminus \partial\Omega$ and $\mathcal{E}_H^b = \mathcal{E}_H \setminus \mathcal{E}_H^0$.

Let Ψ_i $(i = 1, \ldots, N)$ be multiscale basis functions which are derived by solving local problems on each $K \in \mathcal{T}_H$. Define a WG-GMS finite element space V_H as

$$V_H = \text{span}\,\{\Psi_1, \ldots, \Psi_N\}. \tag{2.1}$$

Define a subspace of V_H,

$$V_H^0 = \{v : \ v \in V_H, \ v = 0 \text{ on } \partial\Omega\}. \tag{2.2}$$

Definition 2.1. For any $v \in V_H$ and $K \in \mathcal{T}_H$, let $\nabla_w v$ denote a weak gradient of v on K which should possess the following properties:

P1: $\nabla_w v$ is a good approximation of ∇v on K.
P2: $\nabla_w v$ can be computed effectively.

Definition 2.2. For any $v, w \in V_H$, we define a stabilizer $s(v, w)$ that has the following properties:

P3: $s(v, w)$ is symmetric, semi-positive definite and parameter independent.
P4: $s(v, w) = 0$ if v or w is continuous.

Let $Q_b g$ be an approximation of g. Then we have the WG-GMS finite element method stated as follows.

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